Lecture 18: Second derivatives and concavity. Analysis of polynomial and rational functions.

Victoria LEBED, lebed@maths.tcd.ie

MA1S11A: Calculus with Applications for Scientists

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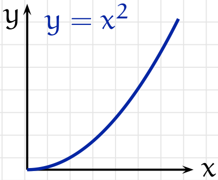
1 Sign of the second derivative

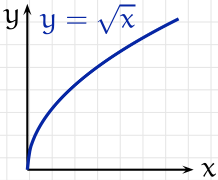
We have seen that f' captures many important properties of f, namely

1) its monotony intervals;
2) its local extrema.

If f is twice differentiable, then f'' can refine the information given by f', as in the second derivative test. But it also encodes some finer properties of f, such as the curvature of its graph.

Example. The graphs of the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$ on $[0, 1]$ share many similarities: both functions are increasing, and take the same values at endpoints.





However, the first graph gets steeper as x increases, and the second one gets less steep. In other words, the graphs are curved differently.

1 Sign of the second derivative

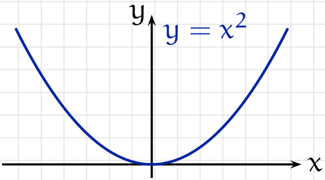
To talk about the curvature rigorously, we need the following terms.

Definition. A function f differentiable on (a, b) is called

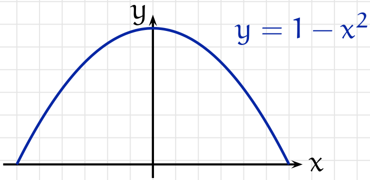
- ✓ **concave up** (or **convex**) if f' is increasing on (a, b);
- ✓ **concave down** (or **concave**) if f' is decreasing on (a, b).

Informally, a function is

- ✓ concave up if it holds water;
- ✓ concave down if it spills water.



concave up



concave down

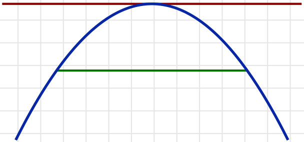
1 Sign of the second derivative

There are two alternative definitions of a concave up/down function:

2) a function lying above/below any of its tangents;

3) a function lying below/above any of its chords.

concave up



concave down

These definitions compare the function analysed with linear functions.

Definition 3) works for not necessarily differentiable functions.

1 Sign of the second derivative

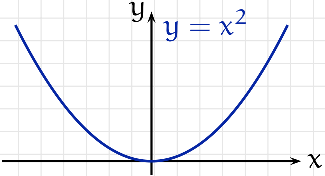
Theorem 7. Let f be a twice differentiable function on (a, b).

(a) If $f''(x) > 0$ on (a, b), then f is concave up.

(b) If $f''(x) < 0$ on (a, b), then f is concave down.

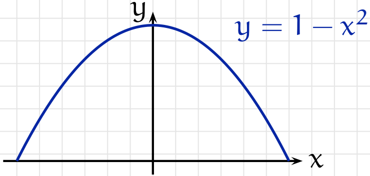
This is true because of the connection between the monotony and the sign of the derivative (Theorem 4), applied to f'.

Example 1.



$f''(x) = 2 > 0$

concave up



$f''(x) = -2 < 0$

concave down

Example 2. For the function $f(x) = x^2 - 6x + 5$, we have $f'(x) = 2x - 6$ and $f''(x) = 2$, so this function is concave up on $\mathbb{R} = (-\infty, +\infty)$.

2 Zeroes of the second derivative

A function seldom has the same concavity type on its whole domain.

Definition. An **inflection point** of a function f is a point where it changes the direction of concavity.

In other words, an inflection point marks the places on the curve $y = f(x)$ where the rate of change of y with respect to x (that is, f') changes from increasing to decreasing, or vice versa.

Theorem 8. If f is a twice differentiable at c, and c is an inflection point for f, then $f''(c) = 0$.

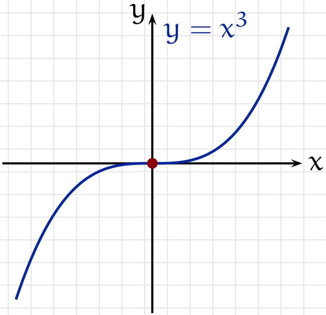
For example, if $f(x)$ is the height of the water level in a vase when it contains x units of water, the inflection points of the graph of $f(x)$ correspond to heights at which the flask is the most narrow, or the most wide.

2 Zeroes of the second derivative

Definition. An **inflection point** of a function f is a point where f changes the direction of concavity.

Theorem 8: c is an inflection point $\implies f''(c) = 0$.

Example 3. For the function $f(x) = x^3$, we have $f'(x) = 3x^2$ and $f''(x) = 6x$, so this function is concave up on $(0, +\infty)$, and is concave down on $(-\infty, 0)$.



$f''(x) < 0$ concave down | $f''(x) = 0$ inflection point | $f''(x) > 0$ concave up

2 Zeroes of the second derivative

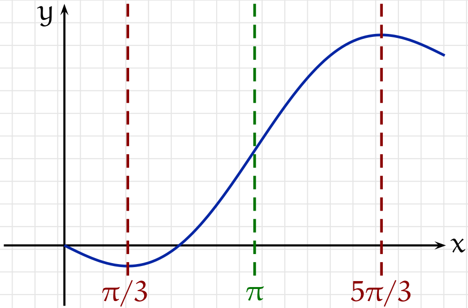
Example 4. Let us analyse the inflection points of the function $f(x) = \frac{x}{2} - \sin x$ on $[0, 2\pi]$ that we considered earlier. We have

$$f'(x) = \frac{1}{2} - \cos x, \qquad f''(x) = \sin x.$$

	0	$(0, \pi)$	π	$(\pi, 2\pi)$	2π
$f''(x) = \sin x$	0	> 0	0	< 0	0
concavity		up	inflection point	down	

2 Zeroes of the second derivative

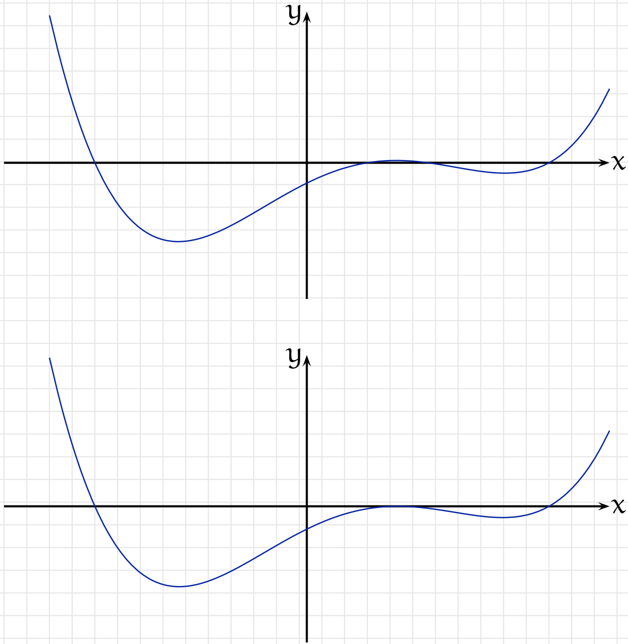
Example 4 (continued). The information gathered about $f(x) = \frac{x}{2} - \sin x$ on $[0, 2\pi]$ (monotony, local and global extrema, behaviour at endpoints, concavity up and down, inflection points) allows us to plot an approximate graph of f:



3 Analysis of polynomial functions

From the viewpoint of calculus, polynomials are the simplest functions:

- ✓ their natural domain is $\mathbb{R}$;
- ✓ they are continuous and differentiable everywhere;
- ✓ they increase or decrease without bound when $x \to \pm\infty$ (depending on the leading term);
- ✓ a polynomial of degree n has at most
  - n roots (i.e., zeroes, or x-intercepts);
  - $n-1$ local extrema;
  - $n-2$ inflection points;
- ✓ a polynomial of odd degree n has no global extrema on $\mathbb{R}$;
- ✓ a polynomial of even degree n has either global maximum but no global minimum on $\mathbb{R}$, or the other way round.


y
x
y
x

3 Analysis of polynomial functions

But polynomials are not entirely boring. Possibly the most interesting part about graphing polynomials is the geometric interpretation of the multiplicities of their roots (which is a priori an algebraic notion):

Definition. Given a polynomial P, a real number r is called a **root of P of multiplicity m** if $(x-r)^m$ divides $P(x)$ but $(x-r)^{m+1}$ does not. In the case $m=1$, r is called a **simple root**.

When we say that a polynomial of degree n has at most n real roots, we count each root with multiplicities.

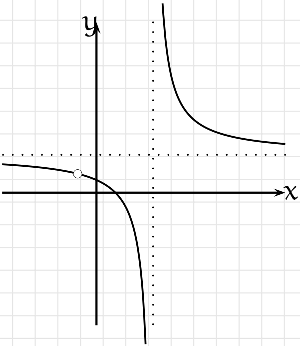
Theorem 8. Suppose that r is a root of $P(x)$ of multiplicity m. Then

parity of m	position of the x-axis w.r.t. the graph of P at $x=r$	inflection point at $x=r$?
even	tangent but not crossing	no
odd >1	tangent and crossing	yes
odd $=1$	not tangent and crossing	yes or no

4 Analysis of rational functions

Analysis of rational functions is more involved, since there are variations in what the domain is, what the asymptotes are etc. The algorithm on the following slides gives a way to identify the most important features of the graph of a rational function, and to sketch it.

We shall deal only with rational functions in the **reduced form** $f(x) = \frac{P(x)}{Q(x)}$, i.e., the polynomials P and Q will have no common factors. If they do, we can cancel all common factors, remember that f is actually not defined where those factors vanish, and remove the respective points from the graph, as we did before.



4 Analysis of rational functions

Definition. Given a rational function in the reduced form $f(x) = P(x)/Q(x)$, a real number r is called

- ✓ a **root of f of multiplicity m** if it is a root of P of multiplicity m.
- ✓ a **pole of f of multiplicity m** if it is a root of Q of multiplicity m.

The behaviour of a rational function close to its roots is the same as for polynomials.

At a pole r of multiplicity m, f has a vertical asymptote, and the sign of f changes at r (from $+\infty$ to $-\infty$ or the other way round) iff m is odd.

4 Analysis of rational functions

Algorithm for graphing $f(x) = P(x)/Q(x)$ (for P and Q without common factors).

- ✓ Determine if the graph has symmetries about the y-axis / the origin, i.e., whether f is even/odd.
- ✓ Find where and how the graph meets the x-axis, i.e., compute the roots of f and their multiplicities.
- ✓ Find where the graph meets the y-axis, i.e., compute $f(0)$.
- ✓ Determine all vertical asymptotes and check if there is a sign change across them, i.e., compute the poles of f and their multiplicities.
- ✓ Compute the limits of $f(x)$ at $\pm\infty$. This yields the horizontal asymptote of the graph, if any.
- ✓ Determine the sign of f on each interval between the x-intercepts and the vertical asymptotes.
- ✓ Determine where f is increasing/decreasing, concave up/down. Find all critical points, local and global extrema, inflection points. For this, analyse the signs of f' and f'', if they exist.
- ✓ Sketch the graph.

5 Analysis of rational functions: example

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

- ✓ *Reduced form:* $\dfrac{x^2-1}{x^3}$,
  since $x^2 - 1 = (x-1)(x+1)$ and x^3 have no common factors.
- ✓ *Symmetries:* $f(-x) = \dfrac{x^2-1}{-x^3} = -\dfrac{x^2-1}{x^3} = -f(x) \implies$ f is odd
  $\implies$ its graph is symmetric about the origin.
- ✓ x-*intercepts:* f has two roots of multiplicity 1: $\pm 1 \implies$ its graph intersects the x-axis at the points ± 1, where the graph changes sign.
- ✓ y-*intercepts:* none, as the point 0 is not in the natural domain of f.
- ✓ *Vertical asymptotes:* f has one pole, 0, of multiplicity 3: $\implies$ at 0 there is a vertical asymptote, and f changes sign.

5 Analysis of rational functions: example

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

- ✓ *Horizontal asymptotes:* We have
$$\lim_{x\to+\infty} \frac{x^2-1}{x^3} = \lim_{x\to+\infty} \left(\frac{1}{x} - \frac{1}{x^3}\right) = 0,$$
$$\lim_{x\to-\infty} \frac{x^2-1}{x^3} = 0 \text{ by symmetry,}$$
so $y = 0$ is the only horizontal asymptote.

- ✓ *Sign analysis:* as seen above, f changes sign at the points $-1, 0, 1$; since $f(2) = \frac{1}{8} > 0$, $f(x) > 0$ on $(1, +\infty)$, which then yields the sign of f everywhere:

interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
sign of f	$-$	$+$	$-$	$+$

5 Analysis of rational functions: example

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

✓ *Monotony and extrema:* the first derivative

$$f'(x) = (x^{-1} - x^{-3})' = -x^{-2} + 3x^{-4} = \frac{3 - x^2}{x^4} = \frac{(\sqrt{3} - x)(x + \sqrt{3})}{x^4}$$

vanishes at $x = \pm\sqrt{3}$; signs of the corresponding factors result in the following signs for f':

interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, +\infty)$
signs of $(\sqrt{3} - x)$ and $(x + \sqrt{3})$	$(+)(-)$	$(+)(+)$	$(+)(+)$	$(-)(+)$
sign of f'	$-$	$+$	$+$	$-$

So, f is decreasing on $(-\infty, -\sqrt{3}]$ and on $[\sqrt{3}, +\infty)$ and is increasing on $[-\sqrt{3}, 0)$ as well as on $(0, \sqrt{3}]$. Therefore, it has a local minimum at $x = -\sqrt{3}$ and a local maximum at $x = \sqrt{3}$.

There is no global minimum or maximum, since $\lim_{x\to 0^+} = -\infty$ and $\lim_{x\to 0^-} = +\infty$.

5 Analysis of rational functions: example

Let us analyse the rational function $f(x) = \frac{x^2-1}{x^3}$.

✓ *Concavity:* the second derivative

$$f''(x) = (f'(x))' = (-x^{-2} + 3x^{-4})' = 2x^{-3} - 12x^{-5}$$
$$= \frac{2(x^2-6)}{x^5} = \frac{2(x-\sqrt{6})(x+\sqrt{6})}{x^5}$$

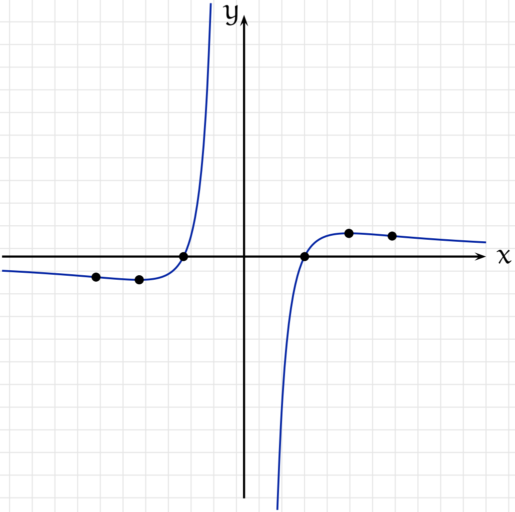
vanishes for $x = \pm\sqrt{6}$;
signs of the corresponding factors result in the following signs for f'':

interval	$(-\infty, -\sqrt{6})$	$(-\sqrt{6}, 0)$	$(0, \sqrt{6})$	$(\sqrt{6}, +\infty)$
signs of factors x^5, $(x-\sqrt{6})$, and $(x+\sqrt{6})$	$(-)(-)(-)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$
sign of f''	$-$	$+$	$-$	$+$

This suggests that f is concave down on $(-\infty, -\sqrt{6}]$ and on $(0, \sqrt{6}]$ and is concave up on $[-\sqrt{6}, 0)$ and on $[\sqrt{6}, +\infty)$.
Therefore, $x = \pm\sqrt{6}$ are inflection points.

Analysis of rational functions: example

Summarising all the available information, we plot the graph of $f(x) = \frac{x^2-1}{x^3}$ as follows:



6 Curvilinear asymptotes

A **curvilinear asymptote** is a curve that approaches a given curve arbitrarily closely.
They are typically used for describing the behaviour of the graph of a function $f(x)$

- ✓ at its poles (this is what vertical asymptotes do);
- ✓ at $\pm\infty$ (this is what horizontal/oblique asymptotes do).

We will now look at a particular type of asymptotes. If for a function $f(x)$ of interest you can find another (preferably simpler) function $g(x)$ such that

$$f(x) - g(x) \underset{x\to+\infty}{\longrightarrow} 0 \qquad \text{or} \qquad f(x) - g(x) \underset{x\to-\infty}{\longrightarrow} 0,$$

then the graph of g is an asymptote of the graph of f. This means that you can understand the behaviour of f for very big/small x by looking at the simpler function g.

We have seen two particular cases of this situation:

- ✓ $g(x) = c$ is a constant function (and defines a horizontal asymptote);
- ✓ $g(x) = bx + c$ is a linear function (and defines an oblique asymptote).

6 Curvilinear asymptotes

To describe a rational function $f(x) = \frac{P(x)}{Q(x)}$ at $\pm\infty$, it is helpful to determine a polynomial function which yields its curvilinear asymptote. For this, you need the polynomial long division. It allows you to write

$$P(x) = Q(x)S(x) + R(x),$$

where S and R are polynomials, and the degree of R is less than the degree of Q. As a result, $\frac{R(x)}{Q(x)} \underset{x\to\pm\infty}{\longrightarrow} 0$ (recall our earlier discussion on the limits of rational functions). So,

$$\frac{P(x)}{Q(x)} - S(x) = \frac{P(x) - Q(x)S(x)}{Q(x)} = \frac{R(x)}{Q(x)} \underset{x\to\pm\infty}{\longrightarrow} 0.$$

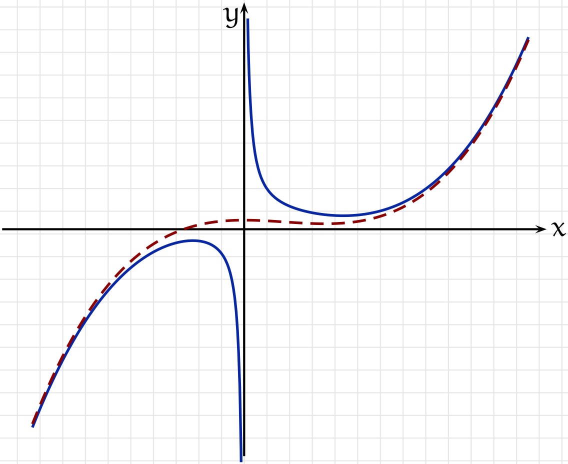
This means that $S(x)$ is the “principal part” of $\frac{P(x)}{Q(x)}$, and the graph of $S(x)$ is a curvilinear asymptote for the graph of $\frac{P(x)}{Q(x)}$.

So, a rational function always has a curvilinear asymptote, and this asymptote is the same at $+\infty$ and at $-\infty$.

6 Curvilinear asymptotes

Example 1. $f(x) = \frac{x^4-2x^3+3x+4}{x} = \frac{x(x^3-2x^2+3)+4}{x} = x^3 - 2x^2 + 3 + \frac{4}{x}$.

So, the graph of $x^3 - 2x^2 + 3$ is an asymptote to the graph of $\frac{x^4-2x^3+3x+4}{x}$.



6 Curvilinear asymptotes

Example 2. $f(x) = \frac{x^4-2}{x^2-1} = \frac{(x^2-1)(x^2+1)-1}{x^2-1} = x^2 + 1 - \frac{1}{x^2-1}$.
So, the graph of $x^2 + 1$ is a parabolic asymptote to the graph of $\frac{x^4-2}{x^2-1}$.