18. More on characters of S_n: Frobenius formula.

Let us summarise what we already know about $\mathrm{Rep}(S_n)$:

1. an explicit construction of Specht reps V_λ, for $\lambda \vdash n$ (in terms of Young polytabloids);
2. a proof that they yield all irreps of S_n;
3. an algorithm for computing the characters of V_λ;
4. recipes for reading information about V_λ off the Young diagram of λ.

We saw complete proofs for 1 - 3, but 4 was given as a series of "spoilers" without proof. Today we will deduce most of those "spoilers" from

Prop. 26 (Frobenius formula): $\forall\, \lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, $\forall\, \lambda' = (\lambda'_1, \dots, \lambda'_{k'}) \vdash n$, $\forall\, \sigma \in S_n$ of cycle type λ',

$$\boxed{\chi^{V_\lambda}(\sigma) = \text{coefficient of } x_1^{\hat\lambda_1} \cdots x_k^{\hat\lambda_k} \text{ in } \underbrace{\prod_{1 \le i < j \le k} (x_i - x_j) \prod_{l=1}^{k'} \left(x_1^{\lambda'_l} + \dots + x_k^{\lambda'_l}\right)}_{=:\, P_{\lambda';k}(x_1, \dots, x_k)}}$$

where $\hat\lambda_k = \lambda_k$, $\hat\lambda_{k-1} = \lambda_{k-1} + 1$, $\hat\lambda_{k-2} = \lambda_{k-2} + 2$, $\dots$, $\hat\lambda_1 = \lambda_1 + (k-1)$.

We will cheat and accept this f-la without proof, since I do not know of a proof elegant enough to be worth sharing. The most curious can have a look at the proof from Fulton–Harris (but be careful, they use a different (but equivalent) construction of the V_λ). However, the deduction of the "spoilers" from Frobenius formula will hopefully give you a feeling of the symmetric polynomial techniques, fundamental in the study of symmetric groups. Another excuse I have is the f-la for $\chi^{\mathbb{C}M_\lambda}(\sigma)$ we proved in Prop. 24: it should make Frobenius f-la less surprising for you.

Thm 12: $\forall \lambda=(\lambda_1,\dots,\lambda_k)\vdash n$,

(1) $\boxed{\iota^*(V_\lambda)=\bigoplus_{\mu\prec\lambda} V_\mu} \in \mathrm{Rep}(S_{n-1})$

here · $\iota: S_{n-1}\to S_n$ is the standard inclusion map

· $\iota^*: \mathrm{Rep}(S_n)\to \mathrm{Rep}(S_{n-1})$ is the induced map

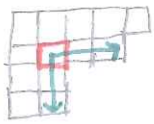
$(V,\rho)\mapsto (V,\rho\circ\iota)$

· $\mu\prec\lambda \Leftrightarrow \lambda=\mu+1_i$ for some $1\le i\le k$

$\Leftrightarrow$ Diag(λ) is obtained from Diag(μ) by adding one cell

(2) $\boxed{\dim_{\mathbb{C}} V_\lambda = \#\, SYT_\lambda}$ (standard Young tableaux of shape λ)

(3) $\boxed{\dim_{\mathbb{C}} V_\lambda = \dfrac{n!}{\prod_{c\in \mathrm{Cell}(D_\lambda)} h(c)}}$ Hook length formula

here $h(c)=$ the hook length of c:  $h(\square)=5$.

We used the following notation, useful in what follows.

For a k-tuple of numbers $\theta=(\theta_1,\dots,\theta_k)$ and any $1\le i\le k$,

- $\theta\pm 1_i := (\theta_1,\dots,\theta_{i-1},\underline{\theta_i\pm 1},\theta_{i+1},\dots,\theta_k)$;
- $\theta\pm 1 := (\theta_1\pm 1,\dots,\theta_k\pm 1)$.

$\square$ (1) Show: $\forall \sigma\in S_{n-1}$, $\chi^{\iota^*(V_\lambda)}(\sigma)=\sum_{\mu\prec\lambda}\chi^{V_\mu}(\sigma)$.

By the def^n of ι^*, $\chi^{\iota^*(V_\lambda)}(\sigma)=\chi^{V_\lambda}(\iota(\sigma))$, and $\iota(\sigma)$ is of cycle type $(\lambda',1)$, where λ' is the cycle type of σ.

so $\chi^{V_\lambda}(\iota(\sigma)) \overset{\text{Frobenius}}{=}$ coeff. of $x_1^{\hat\lambda_1}\cdots x_k^{\hat\lambda_k}$ in $\underbrace{\prod_{1\le i<j\le k}(x_i-x_j)\prod_{e=1}^{k'}(x_1^{\lambda'_e}+\dots+x_k^{\lambda'_e})(x_1+\dots+x_k)}_{=:\,P_{\lambda',k}(x_1,\dots,x_k)}$

$=\sum_{i=1}^{k}\left(\text{coeff. of } \underline{x}^{\hat\lambda-1_i} \text{ in } P_{\lambda',k}(\underline{x})\right)$.

$\{\hat\lambda-1_i \mid 1\le i\le k\}=\{\hat\mu \mid \mu=\lambda-1_i$ is a partition & μ & λ have the same nb of parts$\}$  (1)

$\sqcup\{\hat\lambda-1_k$ having $k-1$ parts$\}$ $\leftarrow$ if $\lambda_k=1$.  (2)

$\sqcup\{$some k-tuples with repeating entries$\}$.  (3)

(3) The polynomials $P_{\lambda',K}$ are anti-symmetric:

$$P_{\lambda',K}(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_K) = -P_{\lambda',K}(x_1,\dots,x_K)$$

$\Rightarrow \forall\, x_1^{\theta_1}\cdots x_K^{\theta_K}$ with $\theta_i=\theta_j$ for some $i\neq j$, its coeff. in $P_{\lambda',K}$ is zero.

(2) Suppose $\lambda_K = 1$, and consider the case $i=K$.

$P_{\lambda',K}(x_1,\dots,x_K) = x_1\cdots x_{K-1}\, P_{\lambda',K-1}(x_1,\dots,x_{K-1})$ + some monomials containing x_K.

So coeff. of $\underline{x}^{\hat\lambda-1_K} = x_1^{\hat\lambda_1}\cdots x_{K-1}^{\hat\lambda_{K-1}}$ in $P_{\lambda',K}(\underline{x})$

$=$ coeff. of $x_1^{\hat\lambda_1-1}\cdots x_{K-1}^{\hat\lambda_{K-1}-1}$ in $P_{\lambda',K-1}(x_1,\dots,x_{K-1})$

$=$ coeff. of $\underline{x}^{\widehat{\lambda-1_K}}$ in $P_{\lambda',K-1}(\underline{x})$,

since $\widehat{\lambda-1_K} = (\lambda_1+(K-2),\dots,\lambda_{K-2}+1,\lambda_{K-1}) = \hat\lambda-1$.

Summarising, $\sum_{i=1}^{K}$ (coeff. of $\underline{x}^{\hat\lambda-1_i}$ in $P_{\lambda',K}(\underline{x})$)

$= \sum_{\mu<\lambda}$ (coeff. of $\underline{x}^{\hat\mu}$ in $P_{\lambda',K(-1)}(\underline{x})$)

$\overset{\text{Frobenius}}{=} \sum_{\mu<\lambda}\chi^{V_\mu}(\sigma)$, as desired.

(2) is proved by induction on n.

- $\dim_{\mathbb{C}} V_1 = 1 = \#SYT_\lambda$
- $\dim_{\mathbb{C}} V_\lambda \overset{(1)}{=} \sum_{\mu<\lambda}\dim_{\mathbb{C}} V_\mu \overset{\text{assumpt}^n}{=} \sum_{\mu<\lambda}\#SYT_\mu \underset{\Uparrow}{=} \#SYT_\lambda$

in a standard Y. tableau, n appears in a removable cell

(3) $\dim_{\mathbb{C}} V_\lambda = \chi^{V_\lambda}(\underset{\uparrow}{Id}) \overset{Fr.}{=}$ coeff. of $\underline{x}^{\hat\lambda}$ in $\prod_{1\le i<j\le K}(x_i-x_j)(x_1+\dots+x_K)^n$.

cycle type 1^n

Vandermonde determinant:

$$\Delta_K(\underline{x}) = \begin{vmatrix} 1 & x_K & x_K^2 & \dots & x_K^{K-1}\\ 1 & x_{K-1} & x_{K-1}^2 & \dots & x_{K-1}^{K-1}\\ \vdots & & & & \\ 1 & x_1 & x_1^2 & \dots & x_1^{K-1}\end{vmatrix} \overset{(*)}{=} \prod_{i<j}(x_i-x_j)$$

$$\underset{\uparrow}{=} \sum_{\sigma\in S_n} sgn(\sigma)\, x_K^{\sigma(1)-1}\cdots x_1^{\sigma(K)-1}$$

by def. of the determinant

Proofs of (**): 1^{st}: $\Delta_K(\underline{x})=0$ when $x_i=x_j$ for some $i\neq j$,

so $\prod_{i<j}(x_i-x_j)$ divides $\Delta_K(\underline{x})$.

Comparing the degrees & the coeff. of $x_{K-1}\dots x_2^{K-2}x_1^{K-1}$,

one concludes $\Delta_K(\underline{x})=\prod_{i<j}(x_i-x_j)$.

2^{nd}: $\prod_{i<j}(x_i-x_j)$ is an anti-symmetric polynomial in $x_1,\dots,x_K$ of degree $\binom{K}{2}$ $\Rightarrow$ as in (3) above, one gets $\prod_{i<j}(x_i-x_j)=\sum_{\sigma\in S_n} a_\sigma x_K^{\sigma(1)-1}\dots x_1^{\sigma(K)-1}$

(other monomials of the right degree have x_i & x_j in the same power for some $i\neq j$).

By antisymmetry again, $a_\sigma=\mathrm{sgn}(\sigma)a_{Id}$ $\forall\sigma\in S_n$.

One concludes by comparing the coeff. of $x_{K-1}\dots x_2^{K-2}x_1^{K-1}$.

So, $\prod_{i<j}(x_i-x_j)(x_1+\dots+x_K)^n=\sum_{\sigma\in S_n}\sum_{n=d_1+\dots+d_K}\mathrm{sgn}(\sigma)\underbrace{\binom{n}{d_1,\dots,d_K}}_{"\frac{n!}{d_1!\dots d_K!}}x_K^{d_1+\sigma(1)-1}\dots x_1^{d_K+\sigma(K)-1}$.

Our $\underline{x}^{\widetilde{\lambda}}$ has in this polynomial the coeff.

$$\sum_{\text{nice}\,\sigma\in S_n}\mathrm{sgn}(\sigma)\frac{n!}{(\widetilde{\lambda}_1-\sigma(K)+1)!\dots(\widetilde{\lambda}_K-\sigma(1)+1)!}, \qquad (*)$$

where "nice" mean satisfying $\widetilde{\lambda}_{K-i+1}-\sigma(i)+1\geq 0$ $\forall\, 1\leq i\leq K$.

$$(*)=\frac{n!}{\widetilde{\lambda}_1!\dots\widetilde{\lambda}_K!}\sum_{\text{all}\,\sigma\in S_n}\mathrm{sgn}(\sigma)\prod_{j=1}^{K}\widetilde{\lambda}_j(\widetilde{\lambda}_j-1)\dots(\widetilde{\lambda}_j-\sigma(K-j+1)+2)$$ { for "bad" σ, the product vanishes

$$=\frac{n!}{\widetilde{\lambda}_1!\dots\widetilde{\lambda}_K!}\begin{vmatrix}1 & \widetilde{\lambda}_K & \widetilde{\lambda}_K(\widetilde{\lambda}_K-1) & \dots\\ \dots & & & \\ 1 & \widetilde{\lambda}_1 & \widetilde{\lambda}_1(\widetilde{\lambda}_1-1) & \dots\end{vmatrix} \underset{\substack{\uparrow\\ \text{column operations}}}{=} \frac{n!}{\widetilde{\lambda}_1!\dots\widetilde{\lambda}_K!}\begin{vmatrix}1 & \widetilde{\lambda}_K & \widetilde{\lambda}_K^2 & \\ \vdots & & & \\ 1 & \widetilde{\lambda}_1 & \widetilde{\lambda}_1^2 & \dots\end{vmatrix} \underset{\substack{\uparrow\\ \text{Vandermonde}}}{=}$$

$$=\frac{n!}{\widetilde{\lambda}_1!\dots\widetilde{\lambda}_K!}\prod_{i<j}(\widetilde{\lambda}_i-\widetilde{\lambda}_j).$$

Conclude by induction, using that $\widetilde{\lambda}_1\dots\widetilde{\lambda}_K=\prod_{c\in 1^{st}\text{ column of }D\lambda}h(c)$. $\boxtimes$