Lecture 17: Derivatives and extrema

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1 Zeroes of the derivative

Definition. Suppose that c is a point from the domain of a function f. We say that f has

- ✓ a **global minimum** at c if $f(c) \leqslant f(x)$ for all x from the domain of f;
- ✓ a **global maximum** at c if $f(c) \geqslant f(x)$ for all x from the domain of f;
- ✓ a **local minimum** at c if $f(c) \leqslant f(x)$ for all x sufficiently close to c;
- ✓ a **local maximum** at c if $f(c) \geqslant f(x)$ for all x sufficiently close to c.

Sometimes the word *global* is replaced with *absolute*, and *local* with *relative*. **Extremum** means minimum or maximum.

Theorem 5. If f is defined on (a, b), differentiable at some $c \in (a, b)$, and attains a (local) extremum at c, then $f'(c) = 0$.

⚠ The converse is false: look again at $f(x) = x^3$ at $x = 0$. We will see later what additional conditions are needed to claim the converse.

1 Zeroes of the derivative

Definition. Let c be a point from the domain of a function f. It is called

- ✓ **critical** if f is not differentiable at c or if $f'(c) = 0$;
- ✓ **stationary** if $f'(c) = 0$.

Theorem 5 says that a function attains local extrema only at critical points:

$$f \text{ has an extremum at } c \quad \Longrightarrow \quad c \text{ is critical.}$$

Example 1. The function $f(x) = |x|$ has a global minimum at $x = 0$. This is a critical point: f is not differentiable at 0.

Example 2. Consider the function $f(x) = x - \sqrt[3]{x}$. We have

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}},$$

so f' is not defined at $x = 0$, and is zero at $x = \pm\frac{1}{\sqrt{27}}$. So, these three points are critical. Stationary points are $\pm\frac{1}{\sqrt{27}}$.

Exercise. Prove that:

- ✓ $x = -\frac{1}{\sqrt{27}}$ is a point of local maximum, and $x = \frac{1}{\sqrt{27}}$ is a point of local minimum; $x = 0$ is not a point of local extremum;
- ✓ f has no global extremum: $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

1 Zeroes of the derivative

Example 3. Let us find all relative extrema of the function $f(x) = x^4 - x^3 + 1$ on $[-1, 1]$. This function is differentiable everywhere, so "suspicious" points are just the stationary points. To determine them, we compute the derivative:

$$f'(x) = 4x^3 - 3x^2.$$

Points c for which $f'(c) = 0$ are $c = 0$ and $c = 3/4$. How to proceed from here?

Let us note that $f'(x) < 0$ for $x < 0$ and $0 < x < 3/4$, and $f'(x) > 0$ for $x > 3/4$. This means that $f(x)$ is decreasing on $[-1, 0]$ and $[0, 3/4]$, and is increasing on $[3/4, 1]$. This in turn means that

- ✓ at $x = 3/4$ a global minimum is attained;
- ✓ at $x = -1$ and $x = 1$ relative maxima are attained;
- ✓ at $x = 0$ we do not have a local extremum at all.

Since $f(-1) = 3 > 1 = f(1)$, f has a global maximum at $x = -1$.

2 First derivative test

First derivative test for local extrema. Suppose that f is differentiable on (a, b) except possibly at c, which is a critical point.

- ✓ If $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then f has a local maximum at c.
- ✓ If $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b), then f has a local minimum at c.
- ✓ If f' has the same sign on (a, c) and on (c, b), then f does not have a local extremum at c.

In practice, one has to look for a sufficiently small interval (a, b) such that f' does not change sign on (a, c) or on (c, b).

2 First derivative test

Example. Let us analyse the critical points of the function $f(x) = x - \sqrt[3]{x}$ we considered earlier. Recall that $f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}}$, so

- ✓ for the stationary point $x = -\frac{1}{\sqrt{27}}$, we have $f'(x) > 0$ on $(a, -\frac{1}{\sqrt{27}})$ and $f'(x) < 0$ on $(-\frac{1}{\sqrt{27}}, b)$ for a and b sufficiently close to $-\frac{1}{\sqrt{27}}$;
- ✓ for the stationary point $x = \frac{1}{\sqrt{27}}$, we have $f'(x) < 0$ on $(a, \frac{1}{\sqrt{27}})$ and $f'(x) > 0$ on $(\frac{1}{\sqrt{27}}, b)$ for a and b sufficiently close to $\frac{1}{\sqrt{27}}$;
- ✓ for the critical point $x = 0$, we have $f'(x) < 0$ for x sufficiently close to 0.

We conclude that f has a relative maximum at $-\frac{1}{\sqrt{27}}$, a relative minimum at $\frac{1}{\sqrt{27}}$, and no relative extremum at 0.

3 Second derivative test

If a function is twice differentiable at a critical point, then a simpler test applies:

Second derivative test for local extrema. Suppose that f is twice differentiable at c, and $f'(c) = 0$.

- ✓ If $f''(c) < 0$, then f has a local maximum at c.
- ✓ If $f''(c) > 0$, then f has a local minimum at c.

If $f''(c) = 0$, then the test is inconclusive: f does not nessecarily have a local extremum at c. In this case, looking at further derivatives might be useful.

Example 1.

$f(x)$	$f'(0)$	$f''(0)$	local extremum at 0?
x^4	0	0	minimum
$-x^4$	0	0	maximum
x^3	0	0	nothing

This example shows that if $f'(c) = f''(c) = 0$, then “anything can happen”.

3 Second derivative test

Example 2. Let us analyse the stationary points of the function $f(x) = \frac{x}{2} - \sin x$ on $[0, 2\pi]$. We have

$$f'(x) = \frac{1}{2} - \cos x, \qquad f''(x) = \sin x.$$

The points c in $[0, 2\pi]$ where f' vanishes are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Substituting into f'', we get

$$f''(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}, \quad f''(\frac{5\pi}{3}) = \sin(\frac{5\pi}{3}) = -\frac{\sqrt{3}}{2}.$$

Conclusion: f has a local maximum at $\frac{5\pi}{3}$, and a local minimum at $\frac{\pi}{3}$.

In practice, we are mostly interested in global, and not local extrema of f. How can one find them?

4 Global extrema

The answer depends on the domain of our function. The simplest domain is a closed interval.

Algorithm for finding global extrema of a function f on $[a, b]$:

1) Determine all critical points of f.
2) Exclude all stationary points which fail first or second derivative tests.
3) Evaluate f at the remaining critical points.
4) Evaluate f at the endpoints a and b.
5) Among the values obtained, choose the minimal and the maximal ones (the same value can be taken at several points).

For a continuous f, the algorithm always gives an answer:

Theorem 6 (Extreme Value Theorem). A continuous function attains a global minimum and a global maximum on any closed interval.

4 Global extrema

Example. We have seen that the function $f(x) = \frac{x}{2} - \sin x$ has two critical points on $[0, 2\pi]$:

1) $x = \frac{5\pi}{3}$ (a local maximum);
2) $x = \frac{\pi}{3}$ (a local minimum).

Let us evaluate f at these points:

1) $f(\frac{5\pi}{3}) = \frac{5\pi}{6} - \sin(\frac{5\pi}{3}) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$;
2) $f(\frac{\pi}{3}) = \frac{\pi}{6} - \sin(\frac{\pi}{3}) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$.

We also need to evaluate f at the endpoints 0 and 2π:

3) $f(0) = \frac{0}{2} - \sin 0 = 0$;
4) $f(2\pi) = \pi - \sin(2\pi) = \pi$.

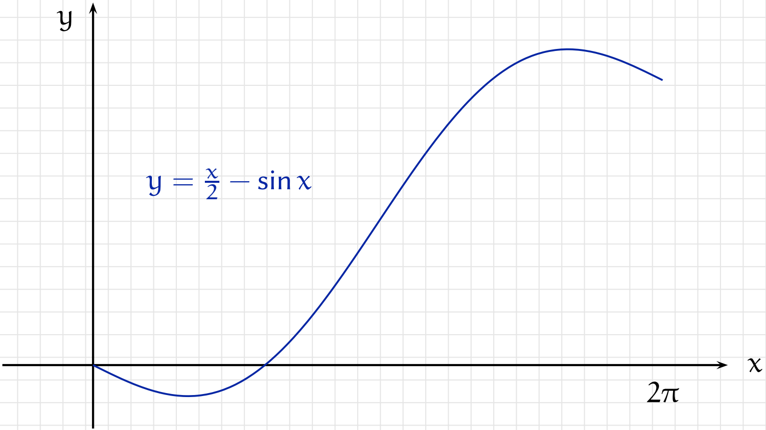
Since $\frac{5\pi}{6} + \frac{\sqrt{3}}{2} > \pi > 0 > \frac{\pi}{6} - \frac{\sqrt{3}}{2}$, we conclude that

1) $\frac{5\pi}{6} + \frac{\sqrt{3}}{2}$ is the maximum of f (attained at $x = \frac{5\pi}{3}$);
2) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$ is the minimum of f (attained at $x = \frac{\pi}{3}$).

4 Global extrema

Example (continued).

1) $\frac{5\pi}{6} + \frac{\sqrt{3}}{2}$ is the maximum of f (attained at $x = \frac{5\pi}{3}$);
2) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$ is the minimum of f (attained at $x = \frac{\pi}{3}$).



4 Global extrema

If the domain is an open interval, things get more complicated, at least because the global extrema need not exist.

Example. The continuous function $f(x) = x$ on $(0, 1)$ or on $\mathbb{R}$ has no extrema. Neither does $f(x) = \tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Algorithm for finding global extrema of f on (a, b) (if they exist):

1) Determine all critical points of f.
2) Exclude all stationary points which fail first or second derivative tests.
3) Evaluate f at the remaining critical points.
4) Compute $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$.
5) Among the values obtained, choose the minimal and the maximal ones. If the minimal/maximal value is one of the limits from 4), and is not attained in any interior point, then f has no global minimum/maximum.

Exercise. Modify the algorithm for half-closed intervals.

In general, the domain of a function f is a disjoint union of intervals. To find global extrema of such f, work separately on every interval from the domain, and then compare the minima/maxima obtained.

4 Global extrema

Example. Let us determine the local and global extrema of

$$f(x) = 2x^3 - 15x^2 + 36x$$

on the half-closed interval $(1, 5]$. We have

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3).$$

Therefore, there are no points of non-differentiability, and the stationary points are $x = 2$ and $x = 3$. Let us apply the 2nd derivative test to them:

$$f'(x) = 12x - 30, \qquad f'(2) = 24 - 30 < 0, \qquad f'(3) = 36 - 30 > 0.$$

So,

- ✓ $x = 2$ is a point of local maximum;
- ✓ $x = 3$ is a point of local minimum.

Finally, we should compare the values of f at the endpoints and at the critical points 2 and 3 that passed the 2nd derivative test.

$$\lim_{x \to 1^+} f(x) = f(1) = 2 - 15 + 36 = 23,$$

$$f(2) = 16 - 60 + 72 = 28,$$

$$f(3) = 54 - 135 + 108 = 27,$$

$$f(5) = 250 - 375 + 180 = 55.$$

Example (continued).

$$\lim_{x\to 1^+} f(x) = f(1) = 2 - 15 + 36 = 23,$$
$$f(2) = 16 - 60 + 72 = 28,$$
$$f(3) = 54 - 135 + 108 = 27,$$
$$f(5) = 250 - 375 + 180 = 55.$$

Since the minimum 23 of these four values is the limit at the endpoint 1 which is excluded from $(1,5]$, our function has no global minimum on $(1,5]$.

On the contrary, the maximum 55 is attained at the endpoint $x = 5$ which is included in $(1,5]$, so it is the point of global maximum.