16. $\mathrm{Irrep}(S_n) = \{\text{Specht reps } V_\lambda \mid \lambda \vdash n\}$

Our today's goal is to prove the statement from the title, which was already illustrated for small n. The proof is decomposed into a list of lemmas.

Recall that a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, written $\lambda \vdash n$, is a decomposition $n = \lambda_1 + \dots + \lambda_k$, $\lambda_1 \geq \lambda_2 \dots \geq \lambda_k > 0$, $\lambda_i \in \mathbb{N}$.
Convention: $\lambda_i = 0$ for $i > k$.

Def.: • $\lambda \trianglerighteq \lambda'$, for $\lambda, \lambda' \vdash n$, means
$$\begin{cases} \lambda_1 \geq \lambda_1' \\ \lambda_1 + \lambda_2 \geq \lambda_1' + \lambda_2' \\ \lambda_1 + \lambda_2 + \lambda_3 \geq \lambda_1' + \lambda_2' + \lambda_3' \\ \dots \end{cases}$$
dominates

• $\lambda \triangleright \lambda'$

Ex: • $(3,1,1) \trianglerighteq (2,2,1)$
• $(4,1,1)$ and $(3,3)$: no domination.

Rmk: Relation $\trianglerighteq$ is a partial order.

Lemma 20: $\lambda, \lambda' \vdash n$, $T \in YT_\lambda$, $T' \in YT_{\lambda'}$.
(a) If all i & j from the same row of T' lie in different columns of T, then $\lambda \trianglerighteq \lambda'$.
(b) If moreover $\lambda = \lambda'$, then $T' \underset{r}{\sim} T'' = \sigma . T$ for some $\sigma \in C_T$.

☐ (a) Put $X'_i = \{ j \mid j \text{ lies in the first } i \text{ rows of } T' \}$. $\# X'_i = \lambda_1' + \dots + \lambda_i'$.
A column of T contains $\leq i$ numbers from X'_i. Move these numbers to the first i rows of T by column permutations.
(Think of the tetris game:

)

But in these i rows there must be $\lambda_1 + \dots + \lambda_i$ numbers. Thus $\lambda_1 + \dots + \lambda_i \geq \lambda_1' + \dots + \lambda_i'$.
(b) Further column permutations of T move all letters from row $p \leq i$ of T' to rows $\leq p$ of T. Since $\lambda_p = \lambda_p'$ for all p, in fact the row p of T' will land precisely inside the row p of T. For $i = \#\mathrm{Rows}(\lambda)$, this yields the desired $T'' \in YT_\lambda$.

☒

Lemma 21: $\lambda, \lambda' \vdash n$, $T \in YT_\lambda$, $T' \in YT_{\lambda'}$.

(a) If $\sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot e_{[T']} \neq 0$, then $\lambda \trianglerighteq \lambda'$.

$\nearrow$ recall the def. of ε_T!

(b) If $\lambda = \lambda'$, then $\boxed{\sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot e_{[T']} = \pm \varepsilon_T \text{ or } 0}$.

☐ (a) To deduce this from L. 20, one needs to show that no i, j from the same row of T' lie in the same column of T. Such i, j would satisfy $(ij) \cdot [T'] = [T']$ & $(ij) \in C_T$. But then $C_T = \{\tau_1, \dots, \tau_\ell, \tau_1(ij), \dots, \tau_\ell(ij)\}$ for some $\tau_k \in S_n$ (coset decomposition).

so, $\sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot e_{[T']} = \sum_{t=1}^{\ell} \left(\mathrm{sgn}(\tau_t) e_{\tau_t \cdot [T']} + \underbrace{\mathrm{sgn}(\tau_t(ij))}_{= -\mathrm{sgn}(\tau_t)} e_{\underbrace{\tau_t(ij) \cdot [T']}_{= \tau_t \cdot [T']}} \right) = 0,$

contradiction.

(b) L. 20 (b) allows one to continue the previous argument if $\lambda = \lambda'$: $\exists\, T' \underset{r}{\sim} T'' = (\sigma')^{-1} \cdot T \Rightarrow$

$$\sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot e_{[T']} \underset{\sigma = \sigma''\sigma'}{\overset{\uparrow}{=}} \sum_{\sigma'' \in C_T} \mathrm{sgn}(\sigma''\sigma')\sigma''\sigma' \cdot e_{[T'']} = \mathrm{sgn}(\sigma') \sum_{\sigma'' \in C_T} \mathrm{sgn}(\sigma'')\sigma'' \cdot e_{[\underbrace{\sigma' \cdot T''}_{= T}]}$$

$$= \mathrm{sgn}(\sigma')\, \varepsilon_T. \quad ☒$$

Since the $e_{[T']}$, $T' \in YT_\lambda$, span $\mathbb{C}M_\lambda$, one gets

Crl 22: $\lambda \vdash n$, $T \in YT_\lambda$, $v \in \mathbb{C}M_\lambda \Rightarrow \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot v = \alpha \varepsilon_T$ for some $\alpha \in \mathbb{C}$.

Lemma 23: $\lambda, \lambda' \vdash n$, $\Phi \in \mathrm{Hom}_{S_n}(\mathbb{C}M_\lambda, \mathbb{C}M_{\lambda'})$, Φ non-zero on $V_\lambda \leq \mathbb{C}M_\lambda$.

(a) Then $\lambda \trianglerighteq \lambda'$.

(b) If moreover $\lambda = \lambda'$, then Φ restricts to $\alpha \, \mathrm{Id}_{V_\lambda}$, for some $\alpha \in \mathbb{C}$.

This statement hopefully recalls you Schur's lemma!

☐ (a) For $T \in YT_\lambda$, $\varepsilon_T \notin \mathrm{Ker}\,\Phi$: otherwise $V_\lambda = \sum_{\sigma \in S_n} \mathbb{C}\sigma \cdot \varepsilon_T \leq \ker \Phi$, since $\ker \Phi$ is a subrep. But Φ is non-zero on V_λ.

So $0 \neq \Phi(\varepsilon_T) = \Phi\left(\sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot e_{[T]}\right) \underset{\substack{\uparrow \\ \Phi \text{ is } \mathbb{C}\text{-}\&\, S_n\text{-linear}}}{=} \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot \Phi(e_{[T]})$

$\Rightarrow 0 \neq \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \cdot e_{[T']}$ for some $T' \in YT_{\lambda'}$

$\Rightarrow \lambda \trianglerighteq \lambda'$, by L. 22.

(b) By Crl 22, $\Phi(\varepsilon_T) = \alpha \varepsilon_T$ for some $\alpha \in \mathbb{C}$. Then $\Phi(\sigma \cdot \varepsilon_T) = \sigma \cdot \Phi(\varepsilon_T) = \alpha \sigma \cdot \varepsilon_T$ for all $\sigma \in S_n$. Thus Φ restricts to $\alpha \, \mathrm{Id}$ on $V_\lambda = \sum_{\sigma \in S_n} \mathbb{C}\sigma \cdot \varepsilon_T$. ☒

Thm 10: (1) $\forall \lambda \vdash n$, V_λ is irreducible

(2) $V_\lambda \cong V_{\lambda'} \Rightarrow \lambda = \lambda'$

(3) $\boxed{\mathrm{Irrep}(S_n) = \{V_\lambda \mid \lambda \vdash n\}}$

(4) $\boxed{\mathbb{C}M_\lambda \cong V_\lambda \oplus \big(\bigoplus_{\lambda' \rhd \lambda} m_{\lambda,\lambda'} V_{\lambda'}\big)}$ for some $m_{\lambda,\lambda'} \in \mathbb{N} \setminus \{0\}$.

☐ (1) Take a subrep. W of $V_\lambda \subseteq \mathbb{C}M_\lambda$, $W \neq \{0\}$.

By Maschke's thm, $\mathbb{C}M_\lambda = W \oplus U$, as S_n-reps.

Consider $\varphi \in \mathrm{Hom}_{S_n}(\mathbb{C}M_\lambda, \mathbb{C}M_\lambda)$, given by $\varphi(w,u) = (w,0)$.

By L.23(b), φ restricts to Id_{V_λ}. But $\varphi(v) = v \Leftrightarrow v \in W$. So $V_\lambda = W$.

(2) $\exists\, \varphi \in \mathrm{Iso}_{S_n}(V_\lambda, V_{\lambda'}) \Rightarrow$ it extends to $\hat{\varphi} \in \mathrm{Hom}_{S_n}(\mathbb{C}M_\lambda, \mathbb{C}M_{\lambda'})$, as in (1) $\underset{\overset{\Uparrow}{L.23.(a)}}{\Longrightarrow} \lambda \trianglerighteq \lambda'$. Starting from φ^{-1}, one gets $\lambda' \trianglerighteq \lambda$.

So $\lambda = \lambda'$.

(3) $\Leftarrow$ (1) & (2) & $\#\mathrm{Irrep}(S_n) = \#\mathrm{Conj}(S_n) \underset{\overset{\Uparrow}{\text{Thm } 7}}{=} \#\{\lambda \vdash n\}$.

(4) By (3), $\mathbb{C}M_\lambda \cong \bigoplus_{\text{all } \lambda' \vdash n} m_{\lambda,\lambda'} V_{\lambda'}$. Now, complete inclusions $V_{\lambda'} \to \mathbb{C}M_\lambda$ to S_n-morphisms $\mathbb{C}M_{\lambda'} \to \mathbb{C}M_\lambda$ as in (1), and use L.23(a). ☒

17. An algorithm for constructing the character table of S_n for $\forall n$.

(4) allows one to construct inductively (w.r.t. $\rhd$) irreps of S_n starting with V_n, using Prop.24.

Ex: $$6 \rhd 5,1 \rhd 4,2 \begin{array}{c} \triangledown\; 3^2 \;\vartriangle \\ \vartriangle\; 4,1^2 \;\triangledown \end{array} 3,2,1 \begin{array}{c} \triangledown\; 2^3 \;\vartriangle \\ \vartriangle\; 3,1^3 \;\triangledown \end{array} 2^2,1^2 \rhd 2,1^4 \rhd 1^6.$$

Prop. 24: $\forall \lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, $\lambda' = (\lambda'_1, \dots, \lambda'_{k'}) \vdash n$, $\sigma \in S_n$ of cycle type λ',

$$\boxed{\chi^{\mathbb{C}M_\lambda}(\sigma) = \text{coefficient of } x_1^{\lambda_1} \dots x_k^{\lambda_k} \text{ in } \prod_{j=1}^{k'} (x_1^{\lambda'_j} + \dots + x_k^{\lambda'_j})}$$

Ex: $n=4$, $\lambda' = (3,1)$, $(x_1^3 + x_2^3)(x_1 + x_2) = x_1^4 + x_2^4 + x_1^3 x_2 + x_2 x_1^3 \Rightarrow$

$$\chi^{\mathbb{C}M_\lambda}((123)) = \begin{cases} 1, & \text{if } \lambda = (4) \text{ or } (3,1), \\ 0, & \text{otherwise}. \end{cases}$$

k=2 suffices
⇑
Cre 25

Rmk: $\prod_j \sum_i x_i^{\lambda'_j}$ is a symmetric polynomial in the x_i.

In fact, the connection between S_n-reps & symmetric polynomials is very deep!

Proof of Prop. 24: $\mathbb{C}M_\lambda$ is $\underline{\underline{a}}$ permutation rep. of S_n.

⚠ It is in general different from $\underline{\underline{\text{the}}}$ permutation rep. $V^{perm} = \bigoplus_{i=1}^{n} \mathbb{C}e_i$, $\sigma\cdot e_i = e_{\sigma(i)}$.

Then $\chi^{\mathbb{C}M_\lambda}(\underset{\uparrow \text{of cycle type } \lambda'}{\sigma}) = \#(M_\lambda)^\sigma = \#\{[T]\in M_\lambda \mid \sigma\cdot[T]=[T]\}$

$= \#\{T\in YT_\lambda \mid \forall i,\ i \,\&\, \sigma(i) \text{ are in the same row of } T\}/\text{row reordering}$

(*)

$= \#\{\varphi : \{1,\dots,k'\}\to\{1,\dots,k\} \mid \forall i,\ \lambda_i = \sum_{\varphi(j)=i}\lambda'_j\}$ ⟵ here $\varphi(j)=i \iff j^{th}$ cycle of σ lies inside i^{th} row of T

$= \#\{\varphi : \{1,\dots,k'\}\to\{1,\dots,k\} \mid \prod_i x_i^{\lambda_i} = \prod_j x_{\varphi(j)}^{\lambda'_j}\}$

$= \sum_\varphi (\text{coef. of } \prod_i x_i^{\lambda_i} \text{ in } \underbrace{\prod_j x_{\varphi(j)}^{\lambda'_j}}_{\text{seen as a polynomial}})$

$= \text{coef. of } \prod_i x_i^{\lambda_i} \text{ in } \sum_\varphi \prod_j x_{\varphi(j)}^{\lambda'_j}$

$= \text{coef. of } \prod_i x_i^{\lambda_i} \text{ in } \prod_j (x_1^{\lambda'_j} + \dots + x_k^{\lambda'_j})$ ☒

Cor 25: $\chi^{\mathbb{C}M_\lambda}(\lambda') \neq 0 \Rightarrow \lambda \trianglerighteq \lambda' \Rightarrow k \leq k'$.

☐ $\exists\, \varphi$ satisfying $(*) \Rightarrow \lambda'_1 \leq \lambda_{\varphi(1)} \leq \lambda_1$,

$\lambda'_1 + \lambda'_2 \leq \underbrace{\lambda_{\varphi(1)} + \lambda_{\varphi(2)}}_{\text{if } \varphi(1)\neq\varphi(2)}$ or $\underbrace{\lambda_{\varphi(1)}}_{\text{if } \varphi(1)=\varphi(2)} \leq \lambda_1 + \lambda_2$,

etc. ☒

So one gets a triangular table

Ex.: $n=4$

	λ'	$1^4 \lhd$	$2,1^2 \lhd$	2^2	$\lhd 3,1$	$\lhd 4$
	$\mathbb{C}M_\lambda \setminus \sigma$	Id	(12)	(12)(34)	(123)	(1234)
$V^{tr} \cong$	$\mathbb{C}M_4$	1	1	1	1	1
$V^{st}\oplus V^{tr} \cong$	$\mathbb{C}M_{3,1}$	4	2	0	1	
$W\oplus V^{st}\oplus V^{tr} \cong$	$\mathbb{C}M_{2^2}$	6	2	2		
$(*) \cong$	$\mathbb{C}M_{2,1^2}$	12	2			
$V^{reg} \cong$	$\mathbb{C}M_{1^4}$	24				

⟸ x_1^4 always appear once

$\mathbb{C}M_{2,1^2} \cong V^{st}\otimes V^{sgn} \oplus W \oplus 2V^{st} \oplus V^{tr}$ (*)

⟸ $= \# S_n$

⇑ $(x_1+\dots+x_4)^4 = \sum \binom{4}{d_1,d_2,d_3,d_4} x_1^{d_1}\dots x_4^{d_4}$ where $\binom{4}{d_1,d_2,\dots} = \frac{4!}{d_1!\,d_2!\dots}$

⇑ $(x_1^2+x_2^2)^2 = x_1^4 + x_2^4 + 2x_1^2x_2^2$

⇑ seen

⇑ $(x_1^2+x_2^2+x_3^2)\underbrace{(x_1+x_2+x_3)^2}_{x_1^2+2x_1x_2+x_2^2+2x_3(x_1+x_2)+x_3^2} = x_1^4 + 2x_1^3x_2 + 2x_1^2x_2^2 + 2x_1^2x_2x_3 + \text{non-relevant terms}$

Our construction of character tables for S_3, S_4, S_5 relied on "happy coincidences": geometric actions, the irreducibility of certain "natural" reps, etc.

But now we have enough tools for designing an

Algorithm for constructing the char. table for any S_n

(1) List all partitions $\lambda \vdash n$.

(2) Order them: $\lambda^{(1)}, \lambda^{(2)}, \ldots$ in such a way that
$\lambda^{(j)} \trianglerighteq \lambda^{(i)} \Rightarrow j \leq i$ (e.g., take the opposite of the lexicographic ordering)

(3) Compute $\chi^{\mathbb{C}M_\lambda}$ for all $\lambda \vdash n$ using Prop. 24.

(4) For $i = 1, 2, \ldots$, in this order, compute

(a) $m_{i,j} = (\chi^{\mathbb{C}M_{\lambda^{(i)}}}, \chi^{V_{\lambda^{(j)}}})$ for all $j < i$.

(b) $\chi^{V_{\lambda^{(i)}}} = \chi^{\mathbb{C}M_{\lambda^{(i)}}} - \sum_{j<i} m_{i,j}\, \chi^{V_{\lambda^{(j)}}}$.

. The computation from the last point is motivated by Thm 10.(4).

This algorithm directly imlies

Thm 11: $\boxed{\text{The characters of all irreps of } S_n \text{ are integer valued: } \chi^{V_\lambda}(\sigma) \in \mathbb{Z}.}$

There are also methods for computing $\chi^{V_i}(\sigma)$ directly:

A) **Frobenius formula:**

$$\chi^{V_\lambda}(\underset{\substack{\uparrow \\ \text{of cycle type } \lambda'}}{\sigma}) = \text{coefficient of } x_1^{\hat{\lambda}_1} \ldots x_k^{\hat{\lambda}_k} \text{ in } \prod_{1 \leq i < j \leq k} (x_i - x_j) \cdot \prod_{j=1}^{k'} (x_1^{\lambda'_j} + \ldots + x_k^{\lambda'_j}).$$

where $\hat{\lambda}_k = \lambda_k, \hat{\lambda}_{k-1} = \lambda_{k-1} + 1, \ldots, \hat{\lambda}_1 = \lambda_1 + k - 1$.

B) **Murnaghan - Nakayama rule**
in terms of skew Young tableaux.

We do not give proofs here.

Rmk: In contrast to what we have seen in the last several lectures, the reps of S_n over a finite field $\mathbb{F}_q$ are not completely understood.