Lectures 14 and 15: Computing derivatives

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MA1S11A: Calculus with Applications for Scientists

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1 Derivative of quotients

Theorem 7. Suppose that the function g is differentiable at $x = x_0$, and that $g(x_0) \neq 0$. Then $\frac{1}{g}$ is differentiable at $x = x_0$, and

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

Proof. We have

$$\begin{aligned}\left(\frac{1}{g}\right)'(x_0) &= \lim_{x\to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x\to x_0} \frac{\frac{g(x_0)-g(x)}{g(x_0)g(x)}}{x - x_0} \\ &= -\frac{1}{g(x_0)} \lim_{x\to x_0} \frac{1}{g(x)} \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = -\frac{g'(x_0)}{g(x_0)^2},\end{aligned}$$

as required. We used that g is differentiable and thus continuous at x_0. $\square$

1 Derivative of quotients

Theorem 8. Suppose that two functions f and g are both differentiable at $x = x_0$, and that $g(x_0) \neq 0$. Then $\frac{f}{g}$ is differentiable at $x = x_0$, and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. Let us apply Theorem 7 and the product rule to the product $f \cdot \frac{1}{g} = \frac{f}{g}$:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x_0) &= f'(x_0)\frac{1}{g(x_0)} + f(x_0)\left(\frac{1}{g}\right)'(x_0) \\ &= \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)g'(x_0)}{g(x_0)^2} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2},\end{aligned}$$

as required. □

1 Derivative of quotients

Example 1.
$$\left(\frac{x^3-2x}{4x^2-3}\right)' = \frac{(x^3-2x)'(4x^2-3)-(x^3-2x)(4x^2-3)'}{(4x^2-3)^2}$$
$$= \frac{(2x^2-2)(4x^2-3)-(x^3-2x)8x}{(4x^2-3)^2}$$
$$= \frac{(8x^4-8x^2-6x^2+6)-(8x^4-16x^2)}{(4x^2-3)^2}$$
$$= \frac{2x^2+6}{(4x^2-3)^2} = 2\frac{x^2+3}{(4x^2-3)^2}.$$

Example 2.
$$\left(\frac{x\sqrt{x}-3x+2}{\sqrt{x}}\right)' = \left(x-3\sqrt{x}+\tfrac{2}{\sqrt{x}}\right)'$$
$$= x' - 3(x^{\frac{1}{2}})' + 2(x^{-\frac{1}{2}})'$$
$$= 1 - \tfrac{3}{2}x^{-\frac{1}{2}} - x^{-\frac{3}{2}}.$$

⚠ Try to simplify a function before computing its derivative!

2 Derivative of trigonometric functions

Theorem 9. We have $\sin'(x) = \cos(x), \qquad \cos'(x) = -\sin(x).$

Proof. We shall use the addition formulas for sines and cosines:
$\sin(a+b) = \sin a \cos b + \cos a \sin b, \quad \cos(a+b) = \cos a \cos b - \sin a \sin b,$

and the remarkable limits $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ (see supplementary materials for details).

Using these formulas, we can evaluate the respective limits:

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \cos x \lim_{h\to 0} \frac{\sin h}{h} - \sin x \lim_{h\to 0} \frac{1-\cos h}{h^2} \lim_{h\to 0} h$$

$$= \cos x \cdot 1 - \sin x \cdot \frac{1}{2} \cdot 0 = \cos x.$$

2 Derivative of trigonometric functions

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Using these formulas, we can evaluate the respective limits:

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} &= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= -\sin x \lim_{h\to 0} \frac{\sin h}{h} - \cos x \lim_{h\to 0} \frac{1-\cos h}{h^2} \lim_{h\to 0} h \\
&= -\sin x \cdot 1 - \cos x \cdot \frac{1}{2} \cdot 0 = -\sin x.
\end{aligned}$$

$\square$

2 Derivative of trigonometric functions

Theorem 10. We have

$$\tan'(x) = \frac{1}{\cos(x)^2} \qquad \text{when } x \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z};$$

$$\cot'(x) = -\frac{1}{\sin(x)^2} \qquad \text{when } x \neq \pi k, k \in \mathbb{Z}.$$

Proof.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}.$$

□

Exercise. Prove the formula for $\cot'$ yourself; this would be an excellent way to start getting used to computing derivatives!

3 Derivative of compositions

Theorem 11 (Chain Rule). Suppose that the function g is differentiable at $x = x_0$, and the function f is differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at x_0, and

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

"Proof." We have

$$\begin{aligned}
\lim_{x \to x_0} \frac{(f \circ g)(x) - (f \circ g)(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} = \\
&= \lim_{x \to x_0} \left[\frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \frac{g(x) - g(x_0)}{x - x_0} \right] = \\
&= \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \\
&= \lim_{t = g(x) \to g(x_0)} \frac{f(t) - f(g(x_0))}{t - g(x_0)} g'(x_0) = f'(g(x_0))g'(x_0).
\end{aligned}$$

□

3 Derivative of compositions

This proof has a gap. The problems lies in the statement

$$\lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0)).$$

For example, for the constant function $g(x) = c$, this becomes $\lim_{x\to x_0} \frac{0}{0} = f'(c)$, which does not make sense!

This problem occurs whenever $g(x)$ is equal to $g(x_0)$ at infinitely many points x that are as close to x_0 as we want.

There are several ways to fix the "proof". One of them uses the ε-δ definition of limits; if you are comfortable with it, try to find a complete proof of the theorem.

Otherwise you have to believe me that the theorem has rigorous proofs. As for the incomplete one we saw above, it can help you memorise the chain rule, and understand why it is natural!

3 Derivative of compositions

Example 1. If $f(x) = \cos(x^3)$, then

$$f'(x) = -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3).$$

If $f(x) = \cos(x)^3$, then
$f'(x) = 3\cos(x)^2 \cdot (-\sin(x)) = -3(1 - \sin(x)^2)\sin(x) = 3\sin(x)^3 - 3\sin(x).$

Example 2. If $f(x) = \tan(x)^2$, then

$$f'(x) = 2\tan(x) \cdot \frac{1}{\cos^2 x} = \frac{2\sin x}{\cos^3 x}.$$

Example 3. If $f(x) = \sqrt{x^2+1}$, then

$$f'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

Example 4. If $g(x) = \frac{1}{f(x)}$ for some function f, then

$$g'(x) = (f(x)^{-1})' = -\frac{1}{f(x)^2} \cdot f'(x) = -\frac{f'(x)}{f(x)^2}.$$

We recover the formula obtained above using the definition of derivatives!

3 Derivative of compositions

With the notation $\frac{dy}{dx}$, the chain rule is the easiest to memorise: if $y = f(g(x))$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

⚠ Recall that $\frac{du}{dx}$ is simply a notation for g'. It is NOT the ration of du and dx, so in the above formula you cannot cancel the two occurrences of du!!!

With the notation $\frac{d}{dx}[f(x)]$, one writes the chain rule as

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}.$$

These forms are useful for integral calculus, namely for the variable change formula.

4 Derivative of inverse functions

Theorem 12. Suppose that the function f is such that:

1. f is invertible;
2. f is continuous on some open interval containing x_0;
3. f is differentiable at $x = x_0$;
4. $f'(x_0) \neq 0$.

Then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$, and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

In other words, $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$.

This theorem, like all the previous ones, has one-sided versions.

Proof. We need to show two things:

A) f^{-1} is differentiable at y_0;

B) $(f^{-1})'(y_0)$ can be computed by the formula above.

Point A) is left for you as an exercise.

Point B) will be proved in two ways.

4 Derivative of inverse functions

The first proof is graphical.

Effect of the reflection about the line $y = x$:

before	after
graph Γ of f point (x_0, y_0) on Γ tangent $y = mx + c$ to Γ at (x_0, y_0) slope $m = f'(x_0)$	graph $\overline{\Gamma}$ of f^{-1} point (y_0, x_0) on $\overline{\Gamma}$ tangent $x = my + c$ to $\overline{\Gamma}$ at (y_0, x_0) slope $\frac{1}{m} = (f^{-1})'(y_0)$

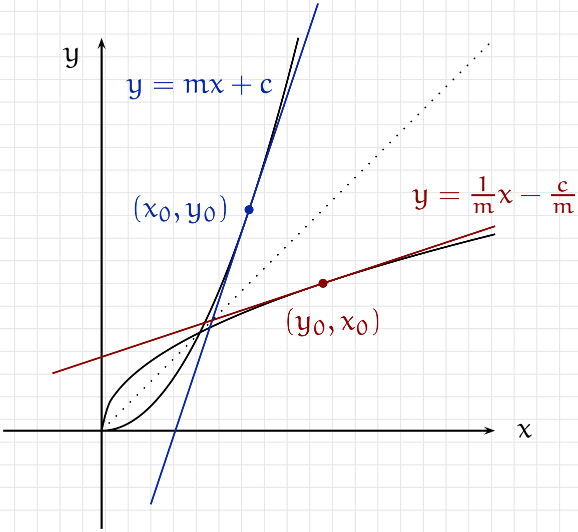
Here $y_0 = f(x_0)$.

To understand the last row,

✓ write the line equation $x = my + c$ in the canonical form: $y = \frac{1}{m}x - \frac{c}{m}$;

✓ recall that the derivative of a function at a point is the slope of the tangent line to its graph at the corresponding point.

4 Derivative of inverse functions



4 Derivative of inverse functions

The second proof is algebraic.

By the definition of the inverse function,

$$f(f^{-1}(x)) = x$$

holds for all x from the range of f.
Differentiation at y_0 yields

$$f'(f^{-1}(y_0))(f^{-1})'(y_0) = 1.$$

Recalling that $f^{-1}(y_0) = x_0$, we get

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)},$$

as desired. $\square$

4 Derivative of inverse functions

Example. Recall that the function $f(x) = \sqrt[n]{x}$ is the inverse of the function $g(x) = x^n$. Here $n \in \mathbb{N}$ (the set of positive integers), and we work with

✓ all x for odd n;

✓ $x \geqslant 0$ for even n.

Then,

$$\sqrt[n]{x}' = f'(x) = \frac{1}{g'(f(x))} = \frac{1}{nf(x)^{n-1}} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{n}x^{-\frac{n-1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

Suppose that $n \geqslant 2$. Then we need to exclude the values of x for which $g'(f(x))$. Since $g'(y) = ny^{n-1}$, this happens when $f(x) = 0$, that is, $x = 0$.

So, we have $\left(x^{\frac{1}{n}}\right)' = \frac{1}{n}x^{\frac{1}{n}-1}$

✓ for $x \neq 0$ and odd n;

✓ for $x > 0$ and even n.

We recovered the formula $(x^r)' = rx^{r-1}$ for $r = \frac{1}{n}$, where $n \in \mathbb{N}$.

Exercise. Prove that $\left(x^{\frac{m}{n}}\right)' = \frac{m}{n}x^{\frac{m}{n}-1}$ for all $x > 0$, $m \in \mathbb{Z}$ (the set of integers), $n \in \mathbb{N}$. (Recall that $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$).

4 Derivative of inverse functions

Example continued. We have established $\left(\sqrt[n]{x}\right)' = \frac{1}{n}x^{\frac{1}{n}-1}$

✓ for $x \neq 0$ and odd n;

✓ for $x > 0$ and even n.

But what happens at $x = 0$ when $n \geqslant 2$?

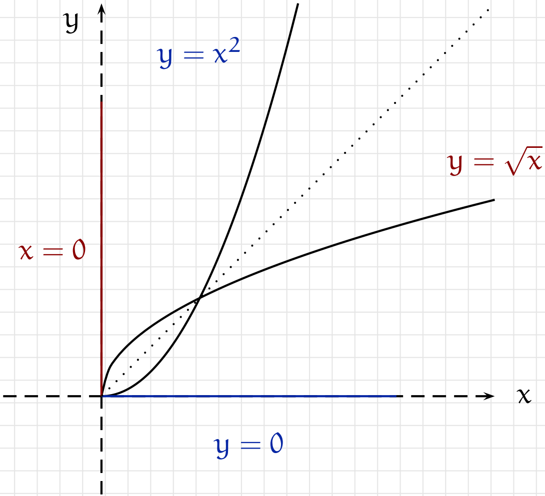
Algebraically, at $x = 0$ we have $\left(\sqrt[n]{x}\right)' = \frac{1}{n}x^{\frac{1}{n}-1} = \dfrac{1}{nx^{1-\frac{1}{n}}} = \frac{1}{0} = \infty$, since $1 - \frac{1}{n} > 0$. So, the function $x^{\frac{1}{n}}$ is not differentiable at 0.

Graphically, since $(x^n)' = nx^{n-1} = 0$ at $x = 0$, the function $g(x) = x^n$ has a horizontal tangent at 0, so its inverse $g^{-1}(x) = \sqrt[n]{x}$ has a vertical tangent at $g(0) = 0$. So, the function $\sqrt[n]{x}$ is not differentiable at 0.

Here for odd n we are talking about two-sided differentiability, and for even n about one-sided differentiability.

4 Derivative of inverse functions

Here is how the tangent lines look like:



5 Derivative of inverse trig. functions

Theorem 13. We have

- ✓ $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$ for all $x \in (1, -1)$;
- ✓ $\arccos'(x) = -\dfrac{1}{\sqrt{1-x^2}}$ for all $x \in (1, -1)$;
- ✓ $\arctan'(x) = \dfrac{1}{1+x^2}$ for all $x \in \mathbb{R}$;
- ✓ $\operatorname{arccot}'(x) = -\dfrac{1}{1+x^2}$ for all $x \in \mathbb{R}$.

These formulas will we particularly useful when computing integrals.

At $x = \pm 1$, the graphs of arcsin and arccos have vertical tangents, so these functions have no one-sided derivatives at $x = \pm 1$.

The proof will be given on the blackboard.

Exercise. Compute $(\arcsin(\sqrt{x}))'$.

6 Summary: Differentiation rules

$$c' = 0, \qquad (x^r)' = rx^{r-1},\ r \in \mathbb{R}$$

$$\sin' = \cos, \qquad \cos' = -\sin$$

$$\tan' = \frac{1}{\cos^2}, \qquad \cot' = -\frac{1}{\sin^2}$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} = -\arccos'(x)$$

$$\arctan'(x) = \frac{1}{1+x^2} = -\operatorname{arccot}'(x)$$

$$(f \pm g)' = f' \pm g', \qquad (cf)' = cf'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}, \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

⚠ While applying these rules, don't forget to indicate for what values of the variable you have the right to do it. That is, specify the domains!

7 Derivative of implicit functions

To compute $(f^{-1})'$, we did not use any explicit formula for f^{-1}. In fact such a formula is rarely available. For instance, what is the inverse function for $f(x) = x^5 + x$?

What we used was the equation $f(f^{-1}(x)) = x$ relating f^{-1} and f.

This is a common situation in practice: we have an equation relating y and x (e.g., a law from physics), using which we want to compute $y'(x)$. In this case we talk about an **implicit function** and **implicit differentiation**. The strategy is to differentiate both sides of the equation, like we did for f^{-1}.

Example 1. To compute $\left(\frac{1}{x}\right)'$ in an alternative way, one can apply implicit differentiation to $xy = 1$:

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1],$$

$$y + xy' = 0,$$

$$y' = -\frac{y}{x} = -\frac{\frac{1}{x}}{x} = -\frac{1}{x^2}.$$

7 Derivative of implicit functions

Example 2. Let us compute $(\sqrt{1-x^2})'$ in two ways.

Method 1: Using the chain rule:

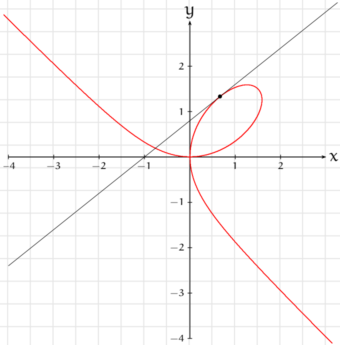
$$\left(\sqrt{1-x^2}\right)' = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

Method 2: Applying implicit differentiation to $x^2 + y^2 = 1$:

$$\begin{aligned} \frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[1], \\ 2x + 2yy' &= 0, \\ yy' &= -x, \\ y' = -\frac{x}{y} &= -\frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

7 Derivative of implicit functions

Example 3. Let us consider the curve $x^3 + y^3 = 3xy$:



What is the equation of the tangent line at the point $(2/3, 4/3)$?
Note that this point is on the curve since

$$(2/3)^3 + (4/3)^3 = 8/27 + 64/27 = 72/27 = 8/3 = 3 \cdot 2/3 \cdot 4/3.$$

7 Derivative of implicit functions

This curve is not a graph of a function, but we still can use derivatives! Differentiating implicitly amounts to the following steps:

$$\frac{d}{dx}[x^3+y^3] = \frac{d}{dx}[3xy],$$
$$3x^2+3y^2y'(x) = 3y+3xy'(x),$$
$$x^2+y^2y'(x) = y+xy'(x),$$
$$(y^2-x)y'(x) = y-x^2,$$
$$y'(x) = \frac{y-x^2}{y^2-x}.$$

Now to compute the slope of the tangent line at a point, we just substitute the x- and y-coordinates. For $(2/3, 4/3)$ we obtain

$$y'(x) = \frac{4/3-4/9}{16/9-2/3} = \frac{8/9}{10/9} = 0.8,$$

and using the point-slope formula, we get $y-4/3 = 0.8(x-2/3)$, that is,

$$y = 0.8x+0.8.$$