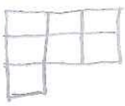
13-14. Representations of S_n & Young Tableaux

We have seen that a finite group G has as many irreps as conjugacy classes. But in general no explicit bijections $\mathrm{Irrep}(G) \xleftrightarrow{1:1} \mathrm{Conj}(G)$ are known.
A pleasant exception is the case of symmetric groups S_n, as we will see today.

Recall (Thm 7): In S_n, $\sigma \sim \sigma' \Leftrightarrow \sigma$ & σ' have the same cycle type.
It is convenient to represent cycle types as
partitions of n: $n = \lambda_1 + \lambda_2 + \ldots + \lambda_k$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$, $\lambda_i \in \mathbb{N}$.

Ex.: $S_7 \ni (725)(143)$ has cycle type $(3,3,1)$, i.e., two 3-cycles (725), (143), and one 1-cycle (6), omitted in its presentation.
The corresponding partition is $7 = 3+3+1$, also written as $\lambda = (3,3,1)$.

A partition can in its turn be represented by
Young diagrams (= Ferrers diagrams):

$7 = 3+3+1 \rightsquigarrow$ 3 3 1  $=: D_{3,3,1} = D_{3^2,1}$

We will learn how to:

① construct irreps out of Young diags: $D_\lambda \rightsquigarrow (V_\lambda, \rho_\lambda) \in \mathrm{Irrep}(S_n)$

② read information about V_λ (such as $\dim_{\mathbb{C}}(V_\lambda)$) off D_λ.

For this we need to upgrade our diagrams:

- **Young tableau** of shape λ := the Young diag. for λ with cells numbered by $1, 2, \ldots, n$;
- a Young tableau is called **standard** if the numbers in each row & each column increase;
- $(S)YT_\lambda = \{$(standard) Young tableaux of shape $\lambda\}$.

Ex.: $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 \\ \cline{1-1}\end{array}$ is a standard Young tableau $\qquad$ $\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & 5 \\ \hline 7 \\ \cline{1-1}\end{array}$ is a non-standard Young tableau

Reps of S_n relate to our combinatorial constructions as follows:

S_n acts on YT_λ by changing labels i to $\sigma(i)$.

Ex.: $(1\,3\,2)\cdot \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & 5 \\ \hline 7 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 3 & 2 & 6 \\ \hline 1 & 4 & 5 \\ \hline 7 \\ \cline{1-1}\end{array}$

On YT_λ, consider the equivalence relation

$T \underset{r}{\sim} T' \Leftrightarrow T'$ is obtained from T by permuting numbers inside the rows.

- **Young tabloid** of shape $\lambda :=$ an equivalence class for $\underset{r}{\sim}$;
- $M_\lambda := \{$Young tabloids of shape $\lambda\} = YT_\lambda / \underset{r}{\sim}$.

Ex.: • $\# M_n = 1$ $\Leftarrow$ all fillings of $\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline\end{array}$ are equivalent

• $\# M_{1,1,\dots,1} = n!$ $\Leftarrow$ all fillings of $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}$ are non-equiv.

} two extreme cases

The above S_n-action is still well defined on M_λ

$\Rightarrow$ one gets a permutation rep. $(\mathbb{C}M_\lambda, \rho_\lambda) \in \mathrm{Rep}(\mathcal{G})$.

This rep. is in general reducible, but has an interesting irred. component. To describe it, we need some more definitions:

- the **column group** of $T \in YT_\lambda$ is

  $C_T := \{\sigma \in S_n \mid \forall i,\ i$ and $\sigma(i)$ are in the same column of $T\}$

  (informally: the subgroup of S_n permuting **numbers** inside the columns of T only);
- the **Young polytabloid** of $T \in YT_\lambda$ is

  $$\boxed{\mathcal{E}_T := \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\,\sigma \cdot e_{[T]} \in \mathbb{C}M_\lambda.}$$

Ex.: • $T = \boxed{1|2|\cdots|n}$, $C_T = \{Id\}$, $\mathcal{E}_T = e_{[T]}$

• $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$, $C_T = S_n$, $\mathcal{E}_T = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) e_{[\sigma \cdot T]}$, where $\sigma \cdot T = \begin{array}{|c|}\hline \sigma(1) \\ \hline \sigma(2) \\ \hline \vdots \\ \hline \sigma(n) \\ \hline \end{array}$

Here $\sigma \cdot T \not\sim_r \sigma' \cdot T$ for $\sigma \neq \sigma'$, so all the $e_{[\sigma \cdot T]}$ are different basis vectors of M_λ.

Lemma 18: $\forall T \in YT_\lambda, \forall \tau \in S_n, \tau \cdot \mathcal{E}_T = \mathcal{E}_{\tau \cdot T}$

$\square$ $\tau \cdot \mathcal{E}_T = \tau \cdot \left(\sum_{\sigma \in C_T} \mathrm{sgn}(\sigma) \sigma \cdot e_{[T]}\right) = \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma) (\tau\sigma) \cdot e_{[T]} =$

$= \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma) (\tau \sigma \tau^{-1}) \cdot (\tau \cdot e_{[T]}) \underset{\substack{\uparrow \\ \sigma' := \tau\sigma\tau^{-1}}}{=} \sum_{\tau^{-1}\sigma'\tau \in C_T} \mathrm{sgn}(\tau^{-1}\sigma'\tau) \sigma' \cdot e_{\tau \cdot [T]} =$

$= \sum_{\sigma' \in C_{\tau \cdot T}} \mathrm{sgn}(\sigma') \sigma' \cdot e_{[\tau \cdot T]} = \mathcal{E}_{\tau \cdot T}$ ⊠.

In the proof, we used

Lemma 19: $C_{\tau \cdot T} = \tau C_T \tau^{-1}$.

$\square$ $\sigma \in C_{\tau \cdot T} \Leftrightarrow \forall$ column of $\tau \cdot T$, its labels form a σ-invariant subset C of $\{1, 2, \ldots, n\}$, called its column filling

$\Leftrightarrow \forall$ column filling C' of T, $\sigma \cdot (\tau \cdot C') = \tau \cdot C'$

$\Leftrightarrow \forall$ column filling C' of T, $(\tau^{-1} \sigma \tau) \cdot C' = C'$

$\Leftrightarrow \tau^{-1} \sigma \tau \in C_T$.

⊠

By Lemma 18, the linear span of the $\mathcal{E}_T$, $T \in YT_\lambda$, is a sub-rep. of $\mathbb{C}M_\lambda$, called the Specht rep. (= Specht module) for λ, and denoted by $\mathbf{V_\lambda} = \underbrace{\sum_{T \in YT_\lambda} \mathbb{C}\mathcal{E}_T}$ $= \sum_{\sigma \in S_n} \mathbb{C}\mathcal{E}_{\sigma \cdot T_0}$. Here T_0 is any tableau from YT_λ

We'll see that the V_λ are irreps, and even more:

Spoiler: $\boxed{\begin{array}{l} (1)\ \mathrm{Irrep}(S_n) = \{V_\lambda \mid \lambda \text{ is a partition of } n\} \\ (2)\ \dim_{\mathbb{C}}(V_\lambda) = \# SYT_\lambda \\ \quad (\text{better: } \mathcal{E}_T, T \in SYT_\lambda, \text{ is a basis of } V_\lambda) \end{array}}$

Ex.: • $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline \end{array}$, $\mathcal{E}_T = e_{[T]}$, $\tau \cdot \mathcal{E}_T = \mathcal{E}_{\tau \cdot T} = \mathcal{E}_T \Leftarrow T \sim_r \tau \cdot T$.

Thus $\boxed{V_n \cong V^{tr}}$.

• $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$, $\mathcal{E}_T = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_{[\sigma \cdot T]}$,

$$\mathcal{E}_{\tau \cdot T} = \tau \cdot \mathcal{E}_T = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_{[(\tau\sigma) \cdot T]} \underset{\underset{\sigma' = \tau\sigma}{\uparrow}}{=} \operatorname{sgn}(\tau) \sum_{\sigma' \in S_n} \operatorname{sgn}(\sigma') e_{[\sigma' \cdot T]} =$$

$$= \operatorname{sgn}(\tau) \cdot \mathcal{E}_T \Rightarrow \underset{\underset{V_{1^n}}{\parallel}}{V_{1,1,\dots,1}} = \sum_{\tau \in S_n} \mathbb{C} \mathcal{E}_{\tau \cdot T} = \mathbb{C} \mathcal{E}_T.$$

Thus $\boxed{V_{1,1,\dots,1} \cong V^{sgn}}$.

In both cases, T is the only standard Young tableau of shape λ, so $\dim_{\mathbb{C}}(V_\lambda) = 1$.

• $M_{2,1} = \left\{ \underset{\parallel\, e_3}{\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \right]}, \underset{\parallel\, e_2}{\left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right]}, \underset{\parallel\, e_1}{\left[\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array} \right]} \right\}$

The number in the lowest cell determines the $\sim_r$ class of the tableau.

$\sigma \cdot [e_i] = [e_{\sigma(i)}]$

$\Rightarrow \mathbb{C} M_{2,1} \cong V^{perm}$, which is reducible

$$\left.\begin{array}{l} C_{e_3} = \{Id, (13)\}, \ \mathcal{E}_{e_3} = e_3 - e_1 \\ C_{e_2} = \{Id, (12)\}, \ \mathcal{E}_{e_2} = e_2 - e_1 \\ C_{e_1} = \{Id, (12)\}, \ \mathcal{E}_{e_1} = e_1 - e_2 \end{array}\right\} \Rightarrow V_{2,1} = \sum_{i=1}^{3} \mathbb{C} \mathcal{E}_{e_i} = \mathbb{C} \mathcal{E}_{e_1} \oplus \mathbb{C} \mathcal{E}_{e_3}$$

is a new realisation of V^{st}.

This argument works in general, and gives (cf. Tutorial 3)

$\mathbb{C} M_{n-1,1} \cong V^{perm}$, $\boxed{V_{n-1,1} \cong V^{st}}$.

Spoiler: (3) $\boxed{V_{\lambda'} \cong V_\lambda \otimes V^{sgn}}$

(4) $S_{n-1} \xrightarrow{\iota} S_n \rightsquigarrow \operatorname{Irrep}(S_n) \xrightarrow{\iota^*} \operatorname{Rep}(S_{n-1})$

$$\boxed{\iota^*(V_\lambda) = \bigoplus_{\mu \prec \lambda} V_\mu}$$

Here: • λ' is the **conjugate partition** for λ: $\lambda = (3,3,1) \Downarrow \lambda' = (3,2,2)$ transpose w.r.t. the diagonal

• ι is the standard inclusion ("act on the first $n-1$ elements")

• ι^* is defined as in Tutorial 1

• $\mu \prec \lambda \Leftrightarrow$ the Young diagram for λ is obtained from that for μ by adding one cell.

Ex: S_4 $\quad \ydiagram{4} \rightsquigarrow V^{tr} \qquad \ydiagram{1,1,1,1} \rightsquigarrow V^{sgn}$

$\ydiagram{3,1} \rightsquigarrow V^{st} \qquad \ydiagram{2,1,1} \rightsquigarrow V^{st} \otimes V^{sgn}$

The only Young diagram whose associated irrep. remains unidentified is $\ydiagram{2,2}$. It has to be the W from Tutorial 2.

(12) yields $\dim_{\mathbb{C}}(\mathcal{V}_{2,2}) = \# SYT_{2,2} = \# \left\{ \young(12,34), \young(13,24) \right\} = 2$

(13) yields $\mathcal{V}_{2,2} \otimes V^{sgn} \cong \mathcal{V}_{2,2}$

(14) yields $i^*(\mathcal{V}_{2,2}) = \mathcal{V}_{2,1}$ $\quad \Leftarrow \ydiagram{2,1} \prec \young(\ \ ,\ \times)$

& no other cells can be removed from $\ydiagram{2,2}$ to get a 3-cell diagram

We thus recover all major properties of W, but also its realisation as a sub-rep. of $\mathbb{C}\mathcal{M}_{2,2}$ of dimension $\binom{4}{2} = 6$ (a basis $\left[\young(ab,cd)\right]$ for $\{a,b,c,d\} = \{1,2,3,4\}$, $a<b$, $c<d$).

Another computation:

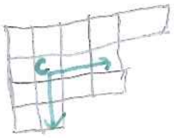
$i^*(S_{3,1}) = S_{2,1} \oplus S_3 \quad \Leftarrow \ydiagram{2,1} \prec \young(\ \ \times,\ )$

$\& \quad \ydiagram{3} \prec \young(\ \ \ ,\times)$

Spoiler: (5) **Hook length formula:**

$$\boxed{\dim(\mathcal{V}_\lambda) = \frac{n!}{\prod_{c \in Cells(D_\lambda)} h(c)}}$$

where $h(c)$ is the **hook length** of the cell c:



$h(c) = 5$

Ex: For $\lambda = (2,2) = (2^2)$, the hook lengths are $\young(32,21)$

$\Rightarrow \dim(\mathcal{V}_{2^2}) = \frac{4!}{3 \cdot 2 \cdot 2 \cdot 1} = 2.$

In practice, the hook length formula is often more efficient than standard tableaux counting.

Ex.: S_5  $\rightsquigarrow V^{tr}$  $\rightsquigarrow V^{sgn}$

 $\rightsquigarrow V^{st}$  $\rightsquigarrow V^{st} \otimes V^{sgn}$

 $\rightsquigarrow V_{3,1^2}$

What do we know about $V := V_{3,1^2}$?

- $\begin{array}{|c|c|c|} \hline 5 & 2 & 1 \\ \hline 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array}$ $\dim_{\mathbb{C}} V = \frac{5!}{5\cdot 2\cdot 2\cdot 1\cdot 1} = 6$

- For $\lambda = (3,1,1)$, $\lambda = \lambda' \Rightarrow V \cong V \otimes V^{sgn}$

Any of these properties is sufficient for deducing

$$\boxed{V_{3,1^2} \cong \Lambda^2(V^{st})}$$

As you probably guess, this relation generalises to all n.

 $\rightsquigarrow V_{3,2} =: R$  $\rightsquigarrow V_{2,2,1} \cong R \otimes V^{sgn}$

$\begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 & \\ \cline{1-2} \end{array}$ $\dim_{\mathbb{C}} R = \frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5.$

$\Rightarrow \underline{V_{3,2} \cong U}$ or $\underline{V_{3,2} \cong U \otimes V^{sgn}}$.

To choose between the two, explicit computations with Young polytabloids are necessary

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Digression: conjugacy classes for S_n.

Thm 9: For a permutation $\sigma \in S_n$ of cycle type $(k^{n_k}, (k-1)^{n_{k-1}}, \ldots, 2^{n_2}, 1^{n_1})$,

$$\boxed{\#[\sigma] = \frac{n!}{k^{n_k} \cdots 2^{n_2}\, n_k! \cdots n_1!}}$$

conjugacy class of σ

Here p^{n_p} stands for $\underbrace{p, p, \ldots, p}_{n_p \text{ times}}$