Lecture 13: Derivatives: properties and computation

Victoria LEBED, lebed@maths.tcd.ie

MA1S11A: Calculus with Applications for Scientists

November 3, 2017

1 Differentiability $\implies$ continuity

We have seen two important properties of a function at a given point: continuity and differentiability. Let us now see how they are related.

Theorem. A function $f(x)$ differentiable at $x = x_0$ is continuous at $x = x_0$.

Proof. We are given that the limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists. To show that f is continuous at $x = x_0$, we need to show that $\lim_{x\to x_0} f(x) = f(x_0)$, or using the new variable $h = x - x_0$, that

$$\lim_{h\to 0}(f(x_0+h) - f(x_0)) = 0.$$

Now we can use arithmetics of limits:

$$\lim_{h\to 0}(f(x_0+h) - f(x_0)) = \lim_{h\to 0}\left[\frac{f(x_0+h) - f(x_0)}{h}\cdot h\right] =$$

$$= \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}\lim_{h\to 0} h = f'(x_0)\cdot 0 = 0.$$

$\square$

2 Differentiability $\not\Leftarrow$ continuity

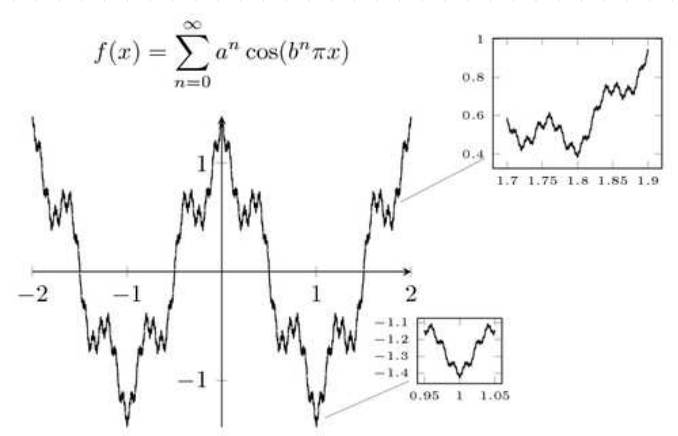
The converse is not true: there exist continuous functions that are not differentiable.

The simplest examples are the functions $f(x) = |x|$ and $f(x) = \sqrt[3]{x}$, which are continuous everywhere but, as we saw, are not differentiable at $x = 0$.

For a long time it was believed that a continuous function would only have a few corner points and points of vertical tangency, with "smooth" pieces between them. Later mathematicians discovered continuous functions that are not differentiable anywhere. A graph of such a function is impossible to plot precisely, but they appear in real life applications, in the context of *fractals* and *Brownian motion of particles.*

2 Differentiability $\not\Leftarrow$ continuity

A famous example of everywhere continuous but nowhere differentiable functions is the *Weierstrass function*:



3 One-sided differentiability

Differentiability, like continuity, has one-sided versions.

Definition. Given a function f, the functions $f'_{\pm}$ defined by the formulas

$$f'_{-}(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h},$$
$$f'_{+}(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

are called the **left-hand** and **right-hand derivatives of f** with respect to x.

Geometrically, $f'_{-}(x)$ is the limit of slopes of secant lines as the point is approached from the left, and $f'_{+}(x)$ is the limit of slopes of secant lines as the point is approached from the right.

Definition. A function defined on a closed interval $[a, b]$ is said to be **differentiable on $[a, b]$** if

- ✓ it is differentiable on the open interval (a, b);
- ✓ the one-sided derivatives $f'_{+}(a)$ and $f'_{-}(b)$ exist.

Differentiability on rays and half-closed intervals is defined similarly.

4 Alternative notations for derivatives

In textbooks, reference manuals etc. you will encounter a variety of notations for derivatives.

Differentiation as operation:

$$f'(x) = \frac{d}{dx}[f(x)], \quad f'(x) = D_x[f(x)], \quad f'(x) = \partial_x[f(x)].$$

Differentiation through the dependent variable: if there is a dependent variable $y = f(x)$, one may use the notation

$$f'(x) = y'(x), \quad f'(x) = \frac{dy}{dx}.$$

Here the fraction $\frac{dy}{dx}$ should not really be viewed as the ratio of two (undefined) quantities dy and dx, but rather as a symbol for the derivative.

4 Alternative notations for derivatives

An evaluation of the derivative at $x = x_0$ in case of a complicated notation is frequently denoted by $\Big|_{x=x_0}$:

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x=x_0},$$

$$f'(x_0) = D_x[f(x)]|_{x=x_0}, \qquad f'(x_0) = \partial_x[f(x)]|_{x=x_0},$$

$$f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\bigg|_{x=x_0}.$$

5 Brick and mortar approach

As usual, to compute the derivatives of basic functions, we will need to:

1) Derive the functions $c, \ x, \ \sin(x)$.

2) Determine how derivation behaves with respect to operations on functions.

Theorem 1. For every $c \in \mathbb{R}$, the constant function $f(x) = c$ is differentiable everywhere, and $(c)' = 0$.

Proof. By the definition of the derivative,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c - c}{x - x_0} = \lim_{x \to x_0} 0 = 0. \qquad \square$$

6 Derivative of power functions

Theorem 2. For every positive integer n, the function $f(x) = x^n$ is differentiable everywhere, and $(x^n)' = nx^{n-1}$.

Proof. For $n = 1$, we have $f(x) = x$ for all x, so

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1 = 1x_0^0.$$

Further, observe that similarly to $a^2 - b^2 = (a-b)(a+b)$, we have

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2), \qquad \ldots$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}),$$

so

$$\lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + xx_0^{n-2} + x_0^{n-1}) = nx_0^{n-1}.$$

□

⚠ This example shows that deriving a product is not as simple as that: $(x \cdot x)' = (x^2)' = 2x$, while $x' \cdot x' = 1 \cdot 1 = 1$. So, $(x \cdot x)' \neq x' \cdot x'$.

6 Derivative of power functions

Theorem 2. For every positive integer n, the function $f(x) = x^n$ is differentiable everywhere, and $(x^n)' = nx^{n-1}$.

We will later prove a more general statement: $(x^r)' = rx^{r-1}$ for all real exponents r and all x for which x^r is defined.

Exercise. Prove the statement for $r = \frac{1}{n}$ and $r = -n$, where n is a positive integer. Use the formula for $a^n - b^n$ above.

Examples.

For $r = \frac{1}{2}$, we get $(x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$ for all $x > 0$. That is, $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.

For $r = -3$, we get $(x^{-3})' = -3x^{-4}$ for all $x \neq 0$. That is, $(\frac{1}{x^3})' = \frac{-3}{x^4}$.

7 Derivative of scalar factors

Contrary to deriving a product, deriving a linear combination of functions is straightforward.

Theorem 3. Suppose that the function f is differentiable at $x = x_0$. Then for any constant c the function cf is differentiable at $x = x_0$, and

$$(cf)'(x_0) = cf'(x_0).$$

Proof. We have

$$(cf)'(x_0) = \lim_{x\to x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{cf(x) - cf(x_0)}{x - x_0} =$$
$$= \lim_{x\to x_0} c\frac{f(x) - f(x_0)}{x - x_0} = c\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = cf'(x_0).$$

□

8 Derivative of sums

Theorem 4. Suppose that two functions f and g are both differentiable at $x = x_0$. Then $f + g$ and $f - g$ are differentiable at $x = x_0$, and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0), \qquad (f-g)'(x_0) = f'(x_0) - g'(x_0).$$

Proof. We have

$$\begin{aligned}
(f+g)'(x_0) = \lim_{x\to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} &= \\
= \lim_{x\to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0} &= \\
= \lim_{x\to x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} &= \\
= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right) &= \\
= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} &= f'(x_0) + g'(x_0).
\end{aligned}$$

The same works for $f - g$. □

9 Derivative of polynomial functions

From the above results, one concludes

Theorem 5. Every polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

is differentiable everywhere, and

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}.$$

Note that the degree of f' is one less the degree of f.

Examples.

$(-x^{18} + \sqrt{2}x^7 - \frac{2}{11}x^2 - 33x + \pi)' = -18x^{17} + 7\sqrt{2}x^6 - \frac{4}{11}x - 33.$

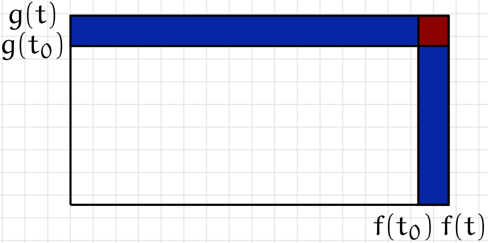
$((x^3 - 1)(2x + 1))' = (2x^4 + x^3 - 2x - 1)' = 8x^3 + 3x^2 - 2.$

10 Derivative of products

Theorem 6 (Product Rule). Suppose that two functions f and g are both differentiable at $x = x_0$. Then fg is differentiable at $x = x_0$, and

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Informally, if f and g describe how two sides of a rectangle change with time t, fg describes the change of the area:



The added area consists of three parts:

✓ the two blue parts correspond to $f'(x_0)g(x_0)$ and $f(x_0)g'(x_0)$;

✓ the red part is relatively small and can be omitted.

10 Derivative of products

Theorem 6 (Product Rule). Suppose that two functions f and g are both differentiable at $x = x_0$. Then fg is differentiable at $x = x_0$, and

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Proof. We have

$$\begin{aligned}
(fg)'(x_0) &= \lim_{x\to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} = \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right) = \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0),
\end{aligned}$$

where we used the fact that a function that is differentiable at x_0 is also continuous at x_0. $\square$

10 Derivative of products

Example 1. Let $f(x) = g(x) = x$, so that $f(x)g(x) = x^2$.
We have $f'(x) = g'(x) = 1$, so

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) = 1 \cdot x + x \cdot 1 = 2x,$$

as expected. In fact, one can use the product rule to get a proof of $(x^n)' = nx^{n-1}$ by induction.

Example 2. $((x^3 - 1)(2x + 1))' = (x^3 - 1)'(2x + 1) + (x^3 - 1)(2x + 1)'$
$= 3x^2(2x + 1) + (x^3 - 1)2 = 6x^3 + 3x^2 + 2x^3 - 2 = 8x^3 + 3x^2 - 2.$
We recover the result obtained by first multiplying $x^3 - 1$ by $2x + 1$.

Example 3. Let $f(x) = (1 + x)\sqrt{x}$. Then

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[1 + x]\sqrt{x} + (1 + x)\frac{d}{dx}[\sqrt{x}] =$$

$$= 1 \cdot \sqrt{x} + (1 + x)\frac{1}{2\sqrt{x}} = \frac{1 + 3x}{2\sqrt{x}}.$$