11-12. Tensor Products of Representations

We have already seen one way of combining two reps V, W into a new one: the direct sum $V \oplus W$. This operation allowed us to reduce the study of the huge set $Rep(G)$ of reps of a finite group G to a finite set $Irrep(G)$.
The aim of today's lecture is to describe a 2nd binary operation on $Rep(G)$, namely, the tensor product $V \otimes W$.
It is of extreme importance in physics, e.g., in quantum mechanics. But for us it will be mainly a source of new irreps: having an incomplete list $V_1, .., V_s$ of irreps of G, new ones can be found by looking at sub-reps of $V_i \otimes V_j$, $1 \le i, j \le s$.
We will use this when studying $Rep(S_5)$.
Our first step will be to define the $\boxed{\text{tensor product of vector spaces}}$. This is a central notion in algebra. Its most basic applications are:

- a description of bilinear maps $V \times W \to U$ as linear maps $V \otimes W \to U$;
- systematic tools for working with multi-dimensional arrays $(\alpha_{i,j,k,\dots})_{\substack{1 \le i \le n_i, \\ 1 \le j \le n_j, \dots}}$, common in physics.
- description of multi-particle systems in quantum mechanics: entangled states $\frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2)$.

Def.: • The **tensor product of vector spaces** $V \& W$ is the quotient space $V \otimes W := \mathbb{C}(V \times W)/_I$, where I is the subspace of $\mathbb{C}(V \times W)$ generated by:

$$\left.\begin{array}{l} \cdot\ (v_1 + v_2, w) - (v_1, w) - (v_2, w) \\ \cdot\ (v, w_1 + w_2) - (v, w_1) - (v, w_2) \\ \cdot\ (\alpha v, w) - \alpha(v, w) \\ \cdot\ (v, \alpha w) - \alpha(v, w) \end{array}\right\} \begin{array}{l} \forall\, v, v_i \in V, \\ w, w_i \in W, \\ \alpha \in \mathbb{C}. \end{array}$$

• The class of $(v, w) \in V \times W$ in $V \otimes W$ is denoted by $v \otimes w$, and called a **pure tensor**.

A general element of $V \otimes W$ has the form $\sum_{i=1}^{n} \alpha_i\, v_i \otimes w_i$ for some $n \in \mathbb{N}$, $v_i \in V$, $w_i \in W$, $\alpha_i \in \mathbb{C}$. The definition of I yields relations $(v_1+v_2)\otimes w = v_1 \otimes w + v_2 \otimes w$, etc.

⚠ Not all the elements of $V \otimes W$ are pure tensors.

Exo: For $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, $e_1 \otimes e_1 + e_2 \otimes e_2$ is not a pure tensor in $V \otimes V$ (i.e., $\neq v \otimes v'$).

We now list the basic properties of tensor products:

(1) an alternative definition in terms of bases (in finite dim.)

(a) $B_V = (e_1, \dots, e_n)$ basis of V,
$B_W = (f_1, \dots, f_m)$ basis of W

$\Downarrow$

$B_V \times B_W = (e_1 \otimes f_1, \dots, e_n \otimes f_1, \dots, e_1 \otimes f_m, \dots, e_n \otimes f_m)$ basis of $V \otimes W$

$\Downarrow$

(b) $\dim_{\mathbb{C}}(V \otimes W) = \dim_{\mathbb{C}}(V)\dim_{\mathbb{C}}(W)$

compare with $V \oplus W$:

$B_V \sqcup B_W = \{e_1, \dots, e_n, f_1, \dots, f_m\}$ is a basis of $V \oplus W$

$\dim_{\mathbb{C}}(V \oplus W) = \dim_{\mathbb{C}}(V) + \dim_{\mathbb{C}}(W)$

(2) arithmetics of vector spaces:

(a) commutativity: $V \otimes W \cong W \otimes V$
(b) associativity: $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$
(c) unit element: $\mathbb{C} \otimes V \cong V \cong V \otimes \mathbb{C}$
(d) distributivity: $(V \oplus W) \otimes U \cong V \otimes U \oplus W \otimes U$
(e) zero element: $0 \otimes V \cong 0 \cong V \otimes 0$

$(V, \otimes, \mathbb{C})$ is a commutative monoid

Rmk: vector spaces with operations $\otimes, \oplus$ exhibit the same properties as non-negative integers $\mathbb{N} \cup \{0\}$, with the usual product & sum operations.

(3) functoriality

Rmk: In Maths this word is used when smth defined at the level of objects of a certain kind (here v. spaces) behaves nicely at the level of morphisms.

(a) $\varphi_i \in \mathrm{Hom}_{\mathbb{C}}(V_i, W_i)$, $i = 1, 2$

$\Rightarrow$ a map $\varphi_1 \otimes \varphi_2 \in \mathrm{Hom}_{\mathbb{C}}(V_1 \otimes V_2, W_1 \otimes W_2)$ can be defined by $\varphi_1 \otimes \varphi_2 (\sum_i \alpha_i v_i^{(1)} \otimes v_i^{(2)}) = \sum_i \alpha_i\, \varphi_1(v_i^{(1)}) \otimes \varphi_2(v_i^{(2)})$.

compare:
$\varphi_1 \oplus \varphi_2 \in \mathrm{Hom}_{\mathbb{C}}(V_1 \oplus V_2, W_1 \oplus W_2)$,
$(\varphi_1 + \varphi_2)(v^{(1)}, v^{(2)}) = (\varphi_1(v^{(1)}), \varphi_2(v^{(2)}))$

(b) If M_i is the matrix of φ_i in bases B_{V_i}, B_{W_i}, then the matrix of $\varphi_1 \otimes \varphi_2$ in bases $B_{V_1} \times B_{V_2}$, $B_{W_1} \times B_{W_2}$ is given by the (non-) Kronecker product of matrices:

$$M_1 \otimes M_2 = \begin{pmatrix} a_{11} M_2 & a_{12} M_2 & \cdots \\ a_{21} M_2 & a_{22} M_2 & \cdots \\ \vdots & & \end{pmatrix} \in Mat_{m_1 m_2, n_1 n_2}(\mathbb{C})$$

$$M_1 \oplus M_2 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \in Mat_{m_1+m_2, n_1+n_2}(\mathbb{C})$$

Here: • $M_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{pmatrix}$

• $n_i = \dim_{\mathbb{C}}(V_i)$, $m_i = \dim_{\mathbb{C}}(W_i)$.

Proof: The block (i,j) of the matrix should correspond to the restriction of $\varphi_1 \otimes \varphi_2$ to $e_j \otimes W_1 \to e'_i \otimes W_2$.
Here $(e_j) = B_{V_1}$, $(e'_i) = B_{V_2}$.
Since $\varphi_1(e_j) = \sum_i a_{ij} e'_i$, this restriction is the map $a_{ij}\varphi_2 : W_1 \to W_2$. ☒

(c) $tr(M_1 \otimes M_2) = tr(M_1)\, tr(M_2)$ if $n_1 = m_1$, $n_2 = m_2$,

$tr(M_1 \oplus M_2) = tr(M_1) + tr(M_2)$

Proof: $tr(M_1 \otimes M_2) = \sum_{i=1}^{n_1} tr(a_{ii} M_2) = \sum_{i=1}^{n_1} (a_{ii}\, tr(M_2)) =$

$= (\sum_{i=1}^{n_1} a_{ii})\, tr(M_2) = tr(M_1)\, tr(M_2)$. ☒

Cor: $tr(\varphi_1 \otimes \varphi_2) = tr(\varphi_1)\, tr(\varphi_2)$

$tr(\varphi_1 \oplus \varphi_2) = tr(\varphi_1) + tr(\varphi_2)$

(d) if $n_1 = m_1$, $n_2 = m_2$, and $v_j^{(i)}$ are the eigenvectors of φ_i, with the eigenvalues $\lambda_j^{(i)}$, then $v_1^{(1)} \otimes v_1^{(2)}, v_2^{(1)} \otimes v_1^{(2)}, \ldots, v_{n_1}^{(1)} \otimes v_{n_2}^{(2)}$ are the eigenvectors of $\varphi_1 \otimes \varphi_2$, with eigenvalues $\lambda_1^{(1)}\lambda_1^{(2)}, \lambda_2^{(1)}\lambda_1^{(2)}, \lambda_3^{(1)}\lambda_1^{(2)}, \ldots, \lambda_{n_1}^{(1)}\lambda_{n_2}^{(2)}$.

$(v_1^{(1)}, 0), \ldots, (v_{n_1}^{(1)}, 0), (0, v_1^{(2)}), \ldots, (0, v_{n_2}^{(2)})$ are the eigenvectors of $\varphi_1 \oplus \varphi_2$, with eigenval. $\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_{n_2}^{(2)}$

Proof: $\varphi_1 \otimes \varphi_2 (v_i^{(1)} \otimes v_j^{(2)}) = \varphi_1(v_i^{(1)}) \otimes \varphi_2(v_j^{(2)}) =$
$= \lambda_i^{(1)} \lambda_j^{(2)}\; v_i^{(1)} \otimes v_j^{(2)}$. ☒

Prop. 15: Take (V, ρ_V) and $(W, \rho_W) \in Rep(G)$. Then:

- $(V \otimes W, \rho_{V\otimes W}) \in Rep(G)$, where $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$
  (i.e., $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$);
- $\chi^{V\otimes W}(g) = \chi^V(g)\,\chi^W(g)$;
- the arithmetic properties (2) (a)-(e) above work at the G-rep. level.

$\square$ Verifications are easy. We'll give details for characters only:

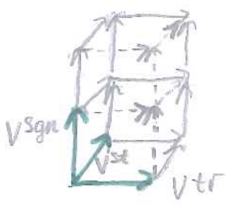
$$\chi^{V\otimes W}(g) = tr(\rho_{V\otimes W}(g)) = tr(\rho_V(g) \otimes \rho_W(g)) \underset{(3)(c)}{=} tr(\rho_V(g))\, tr(\rho_W(g)) = \chi^V(g)\chi^W(g) \boxtimes$$

A memo on characters:

$\chi: Rep(G) \to CF(G)$, a map which is:

- injective
- $Im(\chi) = \bigoplus_{V_i \in Irrep(G)} \mathbb{N}_0 \chi^{V_i}$ is a lattice of max. dim.
- respects the structure:
  - $0 \rightsquigarrow 0$ } constant
  - $V^{tr} \rightsquigarrow 1$ } functions
  - $\oplus \rightsquigarrow +$ } pointwise
  - $\otimes \rightsquigarrow \cdot$ } operations.

Ex.: S_3



Ex.: $G = S_3$,

	1	3	2
	Id	(12)	(123)
V^{tr}	1	1	1
V^{sgn}	1	-1	1
$V := V^{st}$	2	0	-1
$V \otimes V$	4	0	1

$\dim(V \otimes V) = \dim(V)^2 = 4$

$(\chi^{V\otimes V}, \chi^{V^{tr}}) = \frac{1}{6}(1 \cdot 4 \cdot 1 + 3 \cdot 0 \cdot 1 + 2 \cdot 1 \cdot 1) = 1$

$(\chi^{V\otimes V}, \chi^{V^{st}}) = 1$

$(\chi^{V\otimes V}, \chi^{V}) = 1$

$\Rightarrow V \otimes V \cong V^{tr} \oplus V^{sgn} \oplus V$

dim.: $4 = 1 + 1 + 2$

Ex.: $W = \mathbb{C}$, $\rho_W : G \to Aut_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^*$, $V \otimes W \cong V$ as vector spaces;

$\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g) = \underbrace{\rho_W(g)}_{\in \mathbb{C}^*} \underbrace{\rho_V(g)}_{\in Aut_{\mathbb{C}}(V)} \in Aut_{\mathbb{C}}(V)$.

This is the deformation construction you have studied in HW1!
E.g., for $G = S_n$, $V_i^{sgn} \cong V_i \otimes V^{sgn}$ as G-reps.

Now, to construct new irreps out of existing ones, it is common to consider tensor squares $V\otimes V$. Bad news: they are reducible unless $\dim_{\mathbb{C}}(V)=1$ (as we observed for $G=S_3$).

To see this, consider two subspaces of $V\otimes V$:

$S^2(V)=\mathrm{Span}\{v\otimes w+w\otimes v \mid v,w\in V\}$, ← symmetric &

$\Lambda^2(V)=\mathrm{Span}\{v\otimes w-w\otimes v \mid v,w\in V\}$. ← alternating squares of V
(= exterior)

<u>Lemma 16</u>: Let V be a vector space $/\mathbb{C}$, with a basis $(e_1,\dots,e_n)$. Then:

(a)

	basis	dim.
$S^2(V)$	$e_i\otimes e_j+e_j\otimes e_i,\ i\le j$	$\frac{1}{2}n(n+1)$
$\Lambda^2(V)$	$e_i\otimes e_j-e_j\otimes e_i,\ i<j$	$\frac{1}{2}n(n-1)$

(b) $V\otimes V=S^2(V)\oplus\Lambda^2(V)$ $(*)$ (as vector spaces).

□ (a) is straightforward.

(b) • $\dim_{\mathbb{C}}(V\otimes V)=n^2=\dim_{\mathbb{C}}(S^2(V))+\dim_{\mathbb{C}}(\Lambda^2(V))$.

• It remains to check $S^2(V)\cap\Lambda^2(V)=\{0\}$.

Consider the linear map $\tau: V\otimes V\to V\otimes V$, $v\otimes w\mapsto w\otimes v$, called the flip.

For $x\in V\otimes V$, $x\in S^2(V)\Leftrightarrow \tau(x)=x$,

$x\in\Lambda^2(V)\Leftrightarrow\tau(x)=-x$.

So $x\in S^2(V)\cap\Lambda^2(V)\Rightarrow x=\tau(x)=-x\Rightarrow x=0$ ☒

<u>Prop. 17</u>: $V\in\mathrm{Rep}(G)$ (finite) $\Rightarrow$ (a) $S^2(V)$ & $\Lambda^2(V)$ are sub-reps of $V\otimes V$

(b) $(*)$ is a decomposition of G-reps

(c) $\chi^{S^2(V)}(g)=\frac{1}{2}(\chi^V(g)^2+\chi^V(g^2))$

$\chi^{\Lambda^2(V)}(g)=\frac{1}{2}(\chi^V(g)^2-\chi^V(g^2))$

Double-check: $\chi^{V\otimes V}=\chi^{S^2(V)}+\chi^{\Lambda^2(V)}$
$\quad\ \ \| $
$\chi^V\cdot\chi^V$

□ (a) $g\cdot(v\otimes w\pm w\otimes v)=(g\cdot v)\otimes(g\cdot w)\pm(g\cdot w)\otimes(g\cdot v)$

(b) $\Leftarrow$ (a)

(c) Let $(e_1,\dots,e_n)$ be a basis (existence shown in Lecture 4) of eigenvectors for $\rho(g)\in\mathrm{Aut}_{\mathbb{C}}(V)$, with eigenvalues $(\lambda_1,\dots,\lambda_n)$. Then $(e_i\otimes e_j+e_j\otimes e_i)_{i\le j}$ is a basis of eigenvectors for $\rho_{S^2(V)}(g)$, with eigenvalues $\lambda_i\lambda_j$.

So, $\chi^{S^2(V)}(g)=\sum$ eigenvalues of $\rho_{S^2(V)}(g)=\sum_{i\le j}\lambda_i\lambda_j=$

$=\frac{1}{2}\left(\left(\sum_{i=1}^n\lambda_i\right)^2+\sum_{i=1}^n\lambda_i^2\right)=\frac{1}{2}(\chi^V(g)^2+\chi^V(g^2))$.

For $\Lambda^2(V)$, computations are similar.

Ex.: $G = S_3$,

	dim		
$S^2(V)$	3	1	0
$\Lambda^2(V)$	1	-1	1
	$\frac{4\pm 2}{2}$	$\frac{0\pm 2}{2}$	$\frac{1\pm(-1)}{2}$

$\Leftarrow Id^2 = (12)^2 = Id$, $(123)^2 = (132) \in [(123)]$

$\Rightarrow \Lambda^2(V) \cong V^{sgn}$,
$S^2(V) \cong V^{tr} \oplus V$.

Ex.: Character table for S_5

$c \cdot c$	Id	Id	(123)	$(12)(34)$	(12345)	Id	(123)
$\# c$	1	10	20	30	24	15	20
$V \backslash c$	Id	(12)	(123)	(1234)	(12345)	$(12)(34)$	$(12)(345)$
V^{tr}	1	1	1	1	1	1	1
V^{sgn}	1	-1	1	-1	1	1	-1
V^{st}	4	2	1	0	-1	0	-1
$V^{sgn}\otimes V^{st}$	4	-2	1	0	-1	0	1
$\Lambda^2(V^{st})$	6	0	0	0	1	-2	0
U	5	1	-1	-1	0	1	1
$V^{sgn}\otimes U$	5	-1	-1	1	0	1	-1
$\oplus d_i V_i \cong V^{reg}$	120	0	0	0	0	0	0
$V^{tr}\oplus V^{st} \cong V^{perm}$	5	3	2	1	0	1	0
$S^2 \oplus \Lambda^2 \cong V^{st}\otimes V^{st}$	16	4	1	0	1	0	1
$V^{tr}\oplus V^{st}\oplus U \cong S^2(V^{st})$	10	4	1	0	0	2	1

$\} \Sigma = 120$

- $V^{st} \not\cong V^{sgn}\otimes V^{st} \Leftarrow$ different characters.
- $\Lambda^2(V^{st})$ irred. $\Leftarrow (\chi,\chi) = \frac{1}{120}(1\cdot 6^2 + 24\cdot 1^2 + 15\cdot(-2)^2) = 1$.
- computing $\# c$: (1) for $k \le n$, $\#\{k\text{-cycles in } S_n\} = \frac{n(n-1)\dots(n+1-k)}{k}$

  (2) for $k = k_1 + k_2 \le n$, $\#\{\text{perm. of type } (k_1,k_2) \text{ in } S_n\} =$

  $= \frac{n(n-1)\dots(n+1-k)}{k_1 \cdot k_2 \cdot 2^{\delta_{k_1,k_2}}}$
- $V^{sgn}\otimes \Lambda^2(V^{st}) \cong \Lambda^2(V^{st}) \Rightarrow$ no new irrep.
- for $S^2(V^{st})$, $(\chi,\chi) = \frac{1}{120}(1\cdot 10^2 + 10\cdot 4^2 + 20\cdot 1^2 + 15\cdot 2^2 + 20\cdot 1^2) = 3 = \sum m_i^2 \Rightarrow$
  $S^2(V^{st}) \cong V_1 \oplus V_2 \oplus V_3$, V_i distinct irreps

  Using characters: $V_1 = V^{tr}$, $V_2 = V^{st}$, $V_3 =: U$ a new irrep., with $V^{sgn}\otimes U \not\cong U$.