Lecture 12: Basics of differential calculus

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MA1S11A: Calculus with Applications for Scientists

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Recall the three main types of questions asked about functions:

1) When should I have sold my pounds? (extrema)

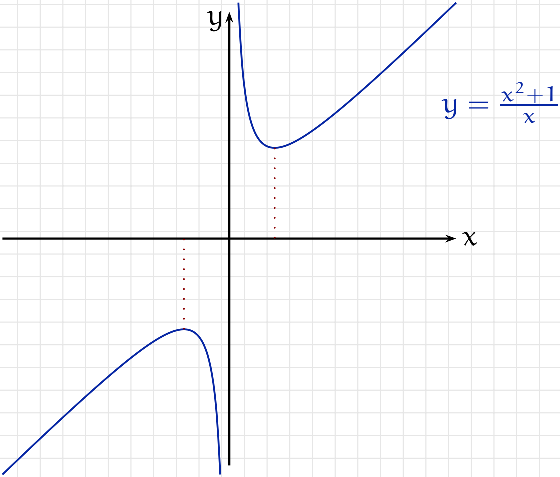
2) How did the exchange rate evolve? (derivative)

3) What is the average rate for the period? (integral)

1 Main questions of calculus

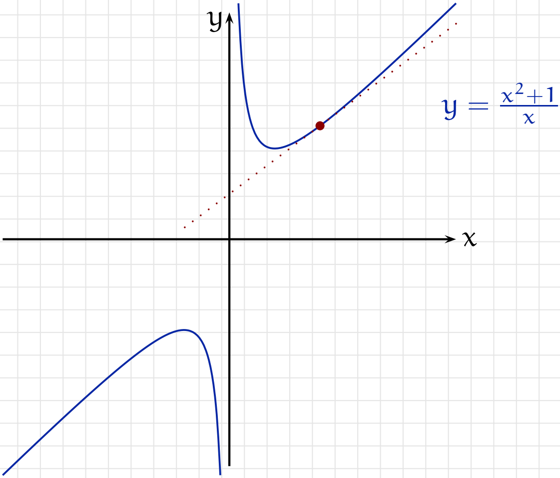
These questions can appear under various disguises:

1) How to determine the points where the values of a function are ("locally") maximal or minimal?



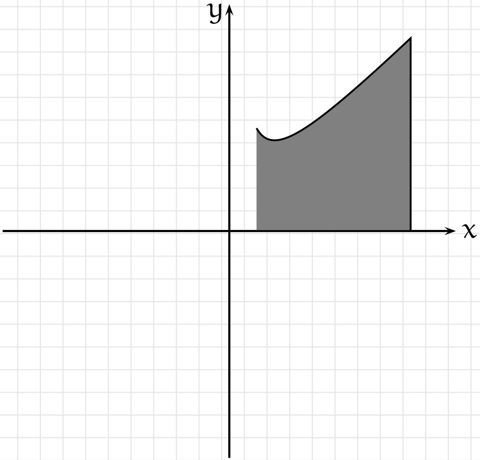
1 Main questions of calculus

2) How to write the equation of a tangent line to a graph?



1 Main questions of calculus

3) How to compute the area under a graph?



1 Main questions of calculus

Questions of types 1) and 2) are dealt with by the differential calculus.
(Relates to the notion of *derivative*.)

Questions of type 3) are dealt with by the integral calculus.
(Relates to the notion of *antiderivative*.)

Calculus = differential calculus + integral calculus.
This is the “practical” part of analysis.

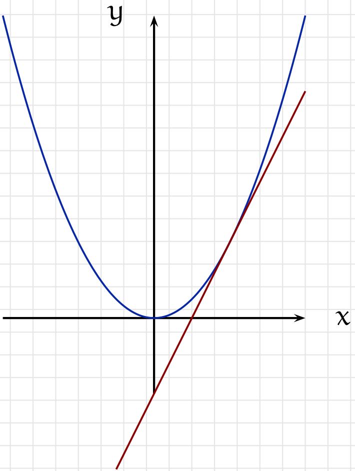
In fact, differential and integral calculi are strongly related by the Newton–Leibniz formula, also called the fundamental theorem of calculus.

These are our topics for the remainder of this module.

2 Derivatives, antiderivative, and limits

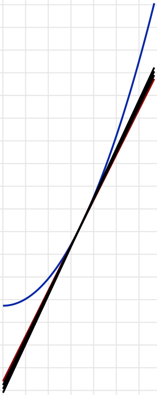
The notion of limit is at the heart of both differential and integral calculi:

2) Look at the tangent line to $y = x^2$ at the point $(1, 1)$:



2 Derivatives, antiderivative, and limits

2) Look at the tangent line to $y = x^2$ at the point $(1, 1)$. It is the limiting position of the secant lines passing through the points $(1, 1)$ and $(1 + h, (1 + h)^2)$ as h gets smaller and smaller:



Here x ranges between 0 and 2.

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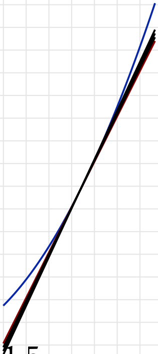

Here x ranges between 0.5 and 1.5.

2 Derivatives, antiderivative, and limits

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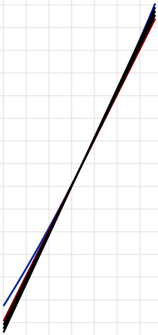
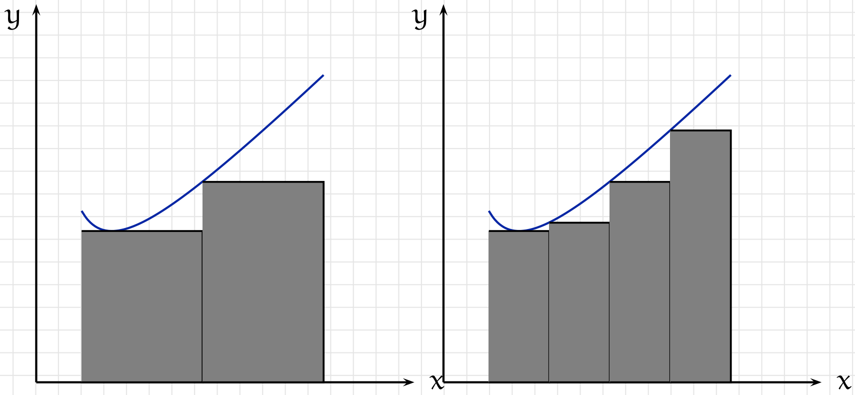


Here x ranges between 0.8 and 1.2.

2 Derivatives, antiderivative, and limits

The notion of limit is at the heart of both differential and integral calculi:

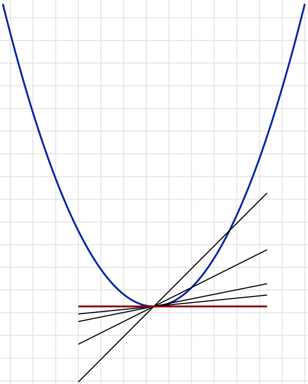
3) We encounter another instance of limiting behaviour when computing areas:



3 The notion of tangent line

We now turn to the differential calculus.

Informal definition. Given a function f, the **tangent line** to its graph at $x = x_0$ is the limit of lines passing through the points $(x_0, f(x_0))$ and $(x, f(x))$ as x approaches x_0.



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Informal definition. Given a function f, the **tangent line** to its graph at $x = x_0$ is the limit of lines passing through the points $(x_0, f(x_0))$ and $(x, f(x))$ as x approaches x_0.

The tangent line is defined by two conditions:

1) it passes through the point $(x_0, f(x_0))$;

2) its slope is the limit of the slopes of lines connecting $(x_0, f(x_0))$ to $(x, f(x))$ as x tends to x_0.

You can think about the tangent line as the best linear approximation of $f(x)$ for x close to x_0.

Linear approximation means approximation by linear functions $ax + b$. Later we will consider approximations by polynomials of higher degrees.

3 The notion of tangent line

Formal definition. Given a function f, the **tangent line** at $x = x_0$ is the line defined by the equation

$$y - f(x_0) = m_{\tan}(x - x_0),$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided that this limit exists.

This definition coincides with the previous one:

1) the line above passes through the point $(x_0, f(x_0))$ since

$$f(x_0) - f(x_0) = 0 = m_{\tan}(x_0 - x_0);$$

2) its slope $m_{\tan}$ is the limit of the slopes $\frac{f(x)-f(x_0)}{x-x_0}$ of lines connecting $(x_0, f(x_0))$ to $(x, f(x))$ as x tends to x_0.

Alternatively, with the change of variables $h = x - x_0$, which is thought as approaching 0, the formula for $m_{\tan}$ becomes

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

4 Tangent line and rate of change

Suppose that $f(x)$ describes the position of a particle moving along the line after x units of time elapsed.

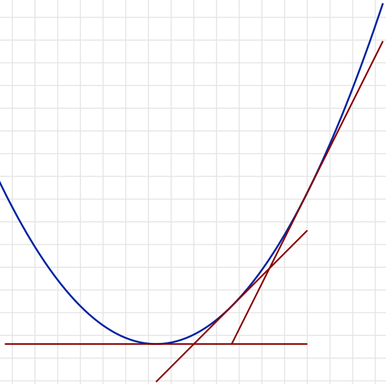
expression	interpretation
$\frac{f(x)-f(x_0)}{x-x_0}$	average velocity on $[x_0, x]$
$m_{\tan} = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$	instantaneous velocity at x_0

More generally, suppose that $f(x)$ describes how a certain quantity f changes depending on a parameter x (e.g. population growth with time, measurements of a metal shape changing depending on temperature, cost change depending on quantity of product manufactured).

expression	interpretation
$\frac{f(x)-f(x_0)}{x-x_0}$	rate of change on $[x_0, x]$
$m_{\tan} = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$	instantaneous rate of change at x_0

5 The notion of derivative

It is important to realise that the limit $m_{\tan}$ we discussed does clearly depend on x_0, so is actually a new function.



5 The notion of derivative

It is important to realise that the limit $m_{\tan}$ we discussed does clearly depend on x_0, so is actually a new function. Let us state clearly its definition (replacing x_0 by x to emphasize the function viewpoint):

Definition. Given a function f, the function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the **derivative of f with respect to x**. The domain of f' consists of all x for which the limit exists.

Definition. A function f is said to be **differentiable at x_0** if the limit $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists.

If f is differentiable at each point of the open interval or ray (a, b), we say that f is **differentiable on (a, b)**.

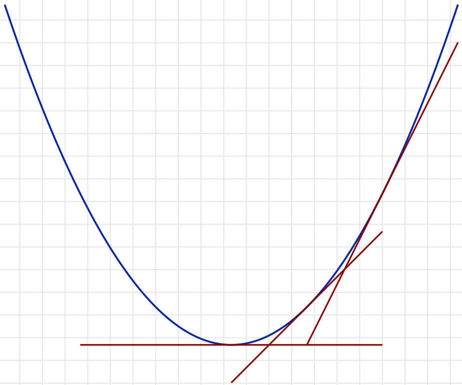
⚠ Here we do not allow infinite limits!

6 Examples

Example 1. Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x.$$

In particular, $f'(0) = 0$, $f'(0.5) = 1$, $f'(1) = 2$.



6 Examples

Example 2. Let $f(x) = \frac{2}{x}$. Then

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{x(x+h)}}{h} =$$

$$= \lim_{h\to 0} \frac{-2h}{hx(x+h)} = \lim_{h\to 0} \frac{-2}{x(x+h)} = -\frac{2}{x^2}.$$

Example 3. Let $f(x) = \sqrt{x}$. Then

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} =$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

7 Algorithm for finding a tangent line

To write the equation of a tangent line to the graph of a function f at $x = x_0$:

1) Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

2) Evaluate $f'(x_0)$ if you can; that is the slope of the tangent line.

3) Write the point–slope equation of the tangent line:

$$y = f'(x_0)(x - x_0) + f(x_0).$$

Example. For $f(x) = x^2$, we have $f'(x_0) = 2x_0$. So, the equation of the tangent line at x_0 is

$$y = 2x_0(x - x_0) + x_0^2.$$

At $x_0 = 0$ this yields $y = 0$.
At $x_0 = 1$ this yields $y = 2(x - 1) + 1$, i.e., $y = 2x - 1$.

8 Points of non-differentiability

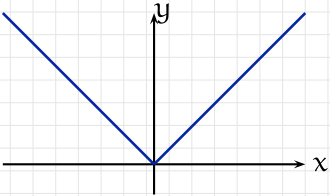
As with continuity, there are various reasons for a function not to be differentiable at a point x_0. Two particularly important situations are:

1) $\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \neq \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$. That is, the growth rate limits on the left and on the right exist but differ. So, there are distinct tangent lines on the left and on the right, forming a "corner point", or a vertex.

Example. Consider the function $f(x) = |x|$ at $x = 0$:

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{x-0}{x} = \lim_{x\to 0^+} 1 = 1,$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{-x-0}{x} = \lim_{x\to 0^-} -1 = -1.$$

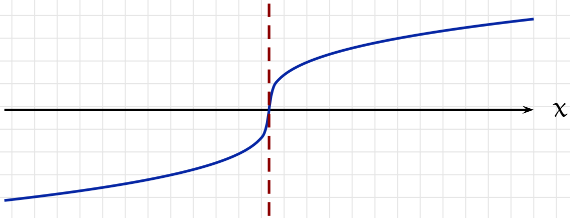


8 Points of non-differentiability

2) $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \pm\infty$. That is, the graph has a vertical tangent line (such a line does not have a finite slope).

Example. Consider the function $f(x) = \sqrt[3]{x}$ at $x = 0$:

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0} \frac{1}{(\sqrt[3]{x})^2} = +\infty.$$



⚠ Geometrically, the function $f(x) = \sqrt[3]{x}$ has a tangent line at $x = 0$, which is vertical. Its equation is $x = 0$. In calculus, only non-vertical tangent line are allowed. Their equations are of the form $y = ax + b$. In this case, f is declared not differentiable at 0.