Lecture 11: Continuity

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MA1S11A: Calculus with Applications for Scientists

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1 The notion of continuity

We have seen that in particularly nice cases (e.g., for polynomials), limits can be computed by evaluation: $\lim_{x\to a} f(x) = f(a)$. To talk about such situations mathematically, we need the following notion.

Definition. A function f is called **continuous at the point $x = a$** if

- ✓ f is defined on an open interval containing $x = a$;
- ✓ $\lim_{x\to a} f(x)$ exists (as a finite number);
- ✓ $\lim_{x\to a} f(x) = f(a)$.

Example. Let us consider the functions

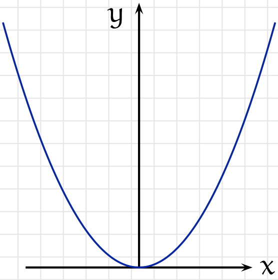
$$f(x) = \frac{x^2-4}{x-2}, \quad g(x) = \begin{cases} \frac{x^2-4}{x-2}, x \neq 2, \\ 3, x = 2, \end{cases} \qquad h(x) = \begin{cases} \frac{x^2-4}{x-2}, x \neq 2, \\ 4, x = 2. \end{cases}$$

These functions all coincide for $x \neq 2$, so they have the same limit at $x \to 2$: $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 4$. The first function is undefined for $x = 2$, so it is not continuous, the second function is defined, but the values do not match, so it is not continuous either, and the third function is continuous.

1 The notion of continuity

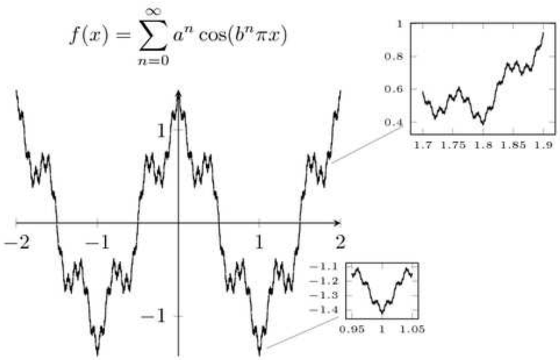
Definition. A function f is called **continuous on an open interval** (a, b) if it is continuous at each point of that interval.

Informally, this means that the graph of f on (a, b) is a single line, drawn without breaks:



1 The notion of continuity

Note that continuous functions can be much more complicated than that:



This is the Weierstrass function. The more you zoom in, the more details of the graph you discern.

1 The notion of continuity

To define continuity on close intervals, we need to deal with continuity at the endpoints:

Definition. A function f is said to be **continuous from the left** (or **from the right**) at the point $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$ (or $\lim_{x \to a^+} f(x) = f(a)$). This means that the limit is defined, the value is defined, and they coincide.

Theorem. A function f is continuous at a iff it is continuous at a both from the left and from the right.

Definition. A function f is called **continuous on a closed interval** $[a, b]$ if

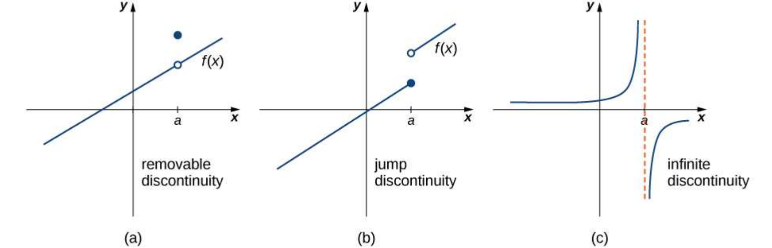
- ✓ f is continuous on (a, b);
- ✓ f is continuous from the right at a;
- ✓ f is continuous from the left at b.

Continuity on half-closed intervals is defined similarly.

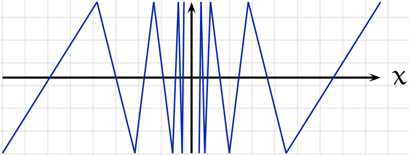
A function f is called **continuous** if its domain is a disjoint union of intervals, and it is continuous on each of these intervals.

2 Discontinuity types

1) removable; 2) jump; 3) infinite;



4) oscillating;

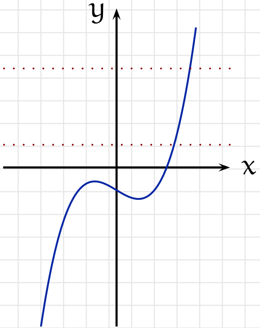


5) mixed.

3 Sneak peek: what is continuity good for?

Intermediate Value Theorem. If a function f is continuous on the closed interval $[a, b]$, then for every value k between $f(a)$ and $f(b)$ the equation $f(x) = k$ has a solution on $[a, b]$.

Example.



For $f(x) = x^3 - x - 1$, we have $f(-2) = -7$, and $f(2.1) = 6.161$, so on the interval $[-2, 2.1]$ this function assumes any value in between, e.g. 1, or $\sqrt{7} + \sqrt[3]{5} \approx 4.356$.

3 Sneak peek: what is continuity good for?

This theorem has several important consequences. One of them is the **bisection method** for numerically solving the equation $f(x) = 0$.
Let us discuss how the method works.
Suppose that f is a continuous function on $[a, b]$, with $f(a) > 0$ and $f(b) < 0$. We want to approximate a zero of f with precision δ.
Compute $f(\frac{a+b}{2})$.

- ✓ If $f(\frac{a+b}{2}) = 0$, then $\frac{a+b}{2}$ is a zero of f.
- ✓ If $f(\frac{a+b}{2}) > 0$, then repeat the method on $[\frac{a+b}{2}, b]$.
- ✓ If $f(\frac{a+b}{2}) < 0$, then repeat the method on $[a, \frac{a+b}{2}]$.

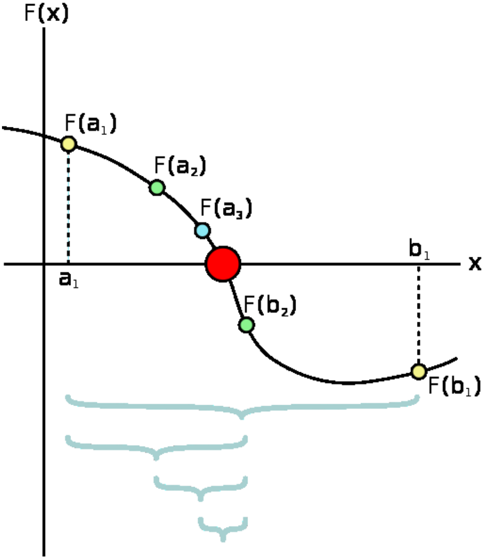
Stop when the length of the interval is $\leqslant 2\delta$, and output its midpoint.
At every iteration,

- ✓ the existence of a zero of f on the new interval is guaranteed by the Intermediate Value Theorem;
- ✓ the search interval decreases by half, so it eventually becomes $\leqslant 2\delta$.

So, the output will be at a distance $\leqslant \delta$ from one of the zeroes of f.

Exercise. Modify the method for functions with $f(a) < 0$, $f(b) > 0$.

Bisection method for numerically solving the equation $F(x) = 0$:



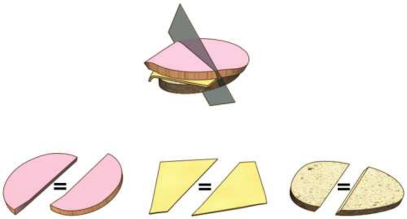
3 Sneak peek: what is continuity good for?

Another application of the Intermediate Value Theorem is the following
Theorem. Whichever shape on the plane (or in the 3d space) we take, there is a straight line (or plane) cutting it into two parts of equal areas (or volumes).

With some work, and still using the IVT, one can prove more:

Pancake theorem. Whichever two shapes on the plane we take, there is a straight line cutting each of them into two parts of the same area.

Ham sandwich theorem. Whichever three shapes in the 3d space we take, there is a plane cutting each of them into two parts of the same volume.



3 Sneak peek: what is continuity good for?

One more application of the Intermediate Value Theorem is the existence of the number $\sqrt{2}$ (which you probably never questioned).

Indeed, the function $f(x) = x^2$ is continuous, and we have

$$f(1) = 1 < 2 < 4 = f(2).$$

Therefore, $f(x) = 2$ for some x between 1 and 2, which is denoted by $\sqrt{2}$.

Applying the bisection method to the function $g(x) = x^2 - 2$, one can approximate $\sqrt{2}$ as precisely as necessary.

4 Continuity on an interval: examples

Example. The function $f(x) = \frac{1}{x}$ is continuous on the closed interval $[2, 3]$. The same function is also continuous on the open interval $(0, 1)$: for every c between 0 and 1 the value $\frac{1}{c}$ is defined, and $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. However, $f(x) = \frac{1}{x}$ is not continuous on the closed interval $[0, 1]$ since $f(0)$ is not defined.

Example. The function $f(x) = \sqrt{9 - x^2}$ is continuous on the closed interval $[-3, 3]$ (its natural domain), since we have

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0 = f(3),$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = 0 = f(-3).$$

5 Proving continuity

We will now show that all basic functions are continuous. As usual, this will be done in two steps:

1) Proving that the functions $c, \ x, \ \sin(x)$ are continuous.

2) Showing that different operations on functions preserve continuity.

Both steps mostly follow from what we already know about limits.

Theorem 1. The functions $c, \ x, \ \sin(x)$ are all continuous on $\mathbb{R}$.

Theorem 2. Suppose that the functions f and g are continuous at a. Then

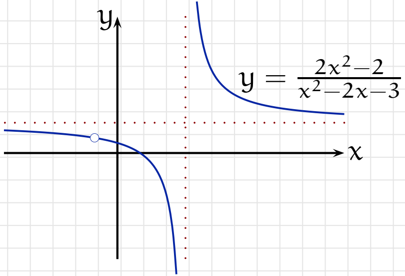
- ✓ the functions $f+g$, $f-g$, fg are all continuous at a;
- ✓ the function f/g is continuous at a if $g(a) \neq 0$, and not continuous (since undefined) at a if $g(a) = 0$;
- ✓ the function $\sqrt[n]{f(x)}$ is continuous at a (for even n, we require $f(x) \geqslant 0$ on some open interval containing a).

5 Proving continuity

Theorem. (Consequence of Theorems 1-2.)

- ✓ A polynomial function is continuous.
- ✓ A rational function is continuous on its domain, i.e., at all points where the denominator is non-zero.

Example. The function $f(x) = \frac{2x^2-2}{x^2-2x-3}$ is continuous at all points where $x^2 - 2x - 3 = (x-3)(x+1) \neq 0$, that is when $x \neq -1$ and $x \neq 3$.



✓ removable discontinuity at -1;

✓ infinite discontinuity at 3.

5 Proving continuity

Theorem 3. Suppose that g is continuous at a, and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.

Theorem 4. Suppose that f is a one-to-one function that is continuous on an interval or a ray. Then

- ✓ f is monotonous;
- ✓ the inverse f^{-1} is continuous on its domain (i.e., on the range of f).

⚠ For the continuity at one point, things are more complicated: f can be continuous at a, without f^{-1} having a limit at $f(a)$.

Theorems 2-4 also work for one-sided continuity.

Example. The polynomial function $f(x) = x^3$ is increasing and continuous on $\mathbb{R}$. So, $f^{-1}(x) = \sqrt[3]{x}$ is continuous on $\mathbb{R}$ (which is the range of f).

Example. The function $f(x) = x^3 + x$ is polynomial and therefore continuous everywhere. It is increasing as a sum of two increasing functions. So, f has a continuous inverse.In this case, it is not easy to describe f^{-1} by a formula, but the theorem still guarantees its continuity!