8-10. Relations between irreps

Our aim is to (finally!) prove the innocent-looking Prop. 7, the deep consequences of which we have been exploiting for quite some time.

Prop. 7: The irreducible characters χ^V, $V \in \mathrm{Irrep}(\mathcal{G})$, form an orthonormal basis of $CF(\mathcal{G})$.

finite group

class functions

Its proof is not straightforward. We will need several preliminary results.

Lemma 10: $V, W \in \mathrm{Rep}(\mathcal{G})$, $\varphi \in \mathrm{Hom}_{\mathcal{G}}(V, W)$ (morphism between $\mathcal{G}$-reps)
$\Rightarrow \mathrm{Ker}\,\varphi \leq V$ and $\mathrm{Im}\,\varphi \leq W$ are sub-reps.

□ $v \in \mathrm{Ker}\,\varphi \Rightarrow \varphi(v) = 0 \Rightarrow \varphi(g \cdot v) = g \cdot \varphi(v) = g \cdot 0 = 0 \Rightarrow g \cdot v \in \mathrm{Ker}\,\varphi$
$w \in \mathrm{Im}\,\varphi \Rightarrow w = \varphi(v)$ for some $v \in V \Rightarrow g \cdot w = g \cdot \varphi(v) = \varphi(g \cdot v) \Rightarrow$
$g \cdot w \in \mathrm{Im}\,\varphi$ ☒

Schur's lemma (1905): $V, W \in \mathrm{Irrep}(\mathcal{G})$, $\varphi \in \mathrm{Hom}_{\mathcal{G}}(V, W) \Rightarrow$
(1) φ is either 0 or an isomorphism
(2) $W = V \Rightarrow \varphi = \lambda \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$.

homothety

This exceptional poverty of homs between irreps will be at the heart of our proof of Prop. 7.

□ (1) By L.10, $\mathrm{Ker}\,\varphi \leq V$ is a sub-rep. Since V is irred., $\mathrm{Ker}\,\varphi$ is either V ($\Rightarrow \varphi = 0$), or 0 ($\Rightarrow \varphi$ injective). In the second case, consider the sub-rep. $\mathrm{Im}\,\varphi \leq W$. It is non-zero, since $V \neq \{0\}$ and φ is injective. But W is irred. Hence $\mathrm{Im}\,\varphi = W$. So φ is surjective, hence bijective.

(2) φ is a lin. transformation from the finite-dim. vector space V over $\mathbb{C}$ to itself $\Rightarrow$ it has an eigenvalue $\lambda \in \mathbb{C}$, with an eigenvector $v \in V$. Then $\varphi - \lambda \mathrm{Id}_V \in \mathrm{End}_G(V)$ in not injective ($\Leftarrow v \in \mathrm{Ker}(\varphi - \lambda \mathrm{Id}_V)$), hence zero (by (1)). So $\varphi = \lambda \mathrm{Id}_V$ $\boxtimes$

Rmk: Our proof for part (1) works for possibly infinite G, and possibly infinite-dim. reps V over any field.
In (2), G can also be infinite, but the field should be algebraically closed, and $\dim_k V < \infty$, since we need the existence of eigenvectors.

Lemma 11: $V, W \in \mathrm{Rep}(G)$, $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V, W) \Rightarrow$

(1) $\boxed{\varphi^{av} := \frac{1}{\#G} \sum_{g \in G} \rho_W(g) \varphi \rho_V(g)^{-1}} \in \mathrm{Hom}_G(V, W)$.

(2) If $V, W \in \mathrm{Irrep}(G)$, $V \not\cong W$, then $\varphi^{av} = 0$.

(3) If $V = W \in \mathrm{Irrep}(G)$, then $\varphi^{av} = \lambda \mathrm{Id}_V$, $\lambda = \frac{\mathrm{tr}(\varphi)}{\dim_{\mathbb{C}} V}$.

(4) $\varphi = \varphi^{av} \Leftrightarrow \varphi \in \mathrm{Hom}_G(V, W)$

We have already encountered this averaging procedure, turning $\mathbb{C}$-linear morphisms into G-linear ones, in the proof of Maschke's theorem.

$\square$ (1) φ^{av} is $\mathbb{C}$-linear since φ and the $\rho_V(g), \rho_W(g)$ are so.
Proof of G-linearity: $\rho_W(h) \varphi^{av} = \frac{1}{\#G} \sum_{g \in G} \rho_W(h) \rho_W(g) \varphi \rho_V(g)^{-1} =$
$= \frac{1}{\#G} \sum_{g \in G} \rho_W(hg) \varphi (\rho_V(hg)^{-1} \rho_V(h)) = \left(\frac{1}{\#G} \sum_{k \in G} \rho_W(k) \varphi \rho_V(k)^{-1}\right) \rho_V(h) =$
$= \varphi^{av} \rho_V(h)$.
We used the variable change $g \rightsquigarrow k = hg$.

(2) $\Leftarrow$ (1) & Schur's lemma.

(3): (1) & Schur's lemma $\Rightarrow \varphi^{av} = \lambda \mathrm{Id}_V$ for some λ.
To determine λ, compute the traces:

$$\left.\begin{aligned} &\mathrm{tr}(\varphi^{av}) = \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}(\rho_V(g) \varphi \rho_V(g)^{-1}) = \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}(\varphi) = \mathrm{tr}(\varphi) \\ &\mathrm{tr}(\lambda \mathrm{Id}_V) = \lambda \dim_{\mathbb{C}}(V) \end{aligned}\right\} \Rightarrow \lambda = \frac{\mathrm{tr}(\varphi)}{\dim_{\mathbb{C}}(V)}$$

(4) • $\varphi = \varphi^{av}$, $\varphi^{av} \in \mathrm{Hom}_G(V, W) \Rightarrow \varphi \in \mathrm{Hom}_G(V, W)$
• $\varphi \in \mathrm{Hom}_G(V, W) \Rightarrow \forall g, \rho_W(g) \varphi \rho_V(g)^{-1} = \varphi \Rightarrow \varphi = \varphi^{av}$ $\boxtimes$

Now choose $V, W \in \mathrm{Irrep}(G)$, and fix their bases.
This allows a translation of Lemma 11 into the matrix language:

<u>lin. transformations</u>	$\rightsquigarrow$ <u>matrices</u>	
$\rho_V(g)$	$R_{ij}(g)$	$1 \le i,j \le \dim_{\mathbb{C}} V =: n$
$\rho_W(g)$	$R'_{ij}(g)$	$1 \le i,j \le \dim_{\mathbb{C}} W =: n'$
$\varphi \in \mathrm{Hom}_{\mathbb{C}}(V,W)$	F_{ij}	$1 \le i \le n'$, $1 \le j \le n$
φ^{av}	$\frac{1}{\#G} \sum_{g \in G} R'(g) F R(g)^{-1}$	

<u>Case $V \not\cong W$</u>: $\forall \varphi$, $\varphi^{av} = 0 \Rightarrow \forall F \in \mathrm{Mat}_{n \times n'}(\mathbb{C})$, $\sum_{g \in G} R'(g) F R(g)^{-1} = 0$

$\Rightarrow \forall\, 1 \le i_1 \le n', 1 \le j_1 \le n$, $\sum_{g \in G} \sum_{\substack{1 \le i_2 \le n' \\ 1 \le j_2 \le n}} R'_{i_1 i_2}(g) F_{i_2 j_2} R_{j_2 j_1}(g^{-1}) = 0$.

$\uparrow$
consider each cell (i_1, j_1)

We get lin. combinations of the $F_{i_2 j_2}$ which vanish for all $F_{i_2, j_2} \in \mathbb{C}$. Thus their coefficients have to vanish:

$\forall\, 1 \le i_1, i_2 \le n'$, $\forall\, 1 \le j_1, j_2 \le n$, $\boxed{\sum_{g \in G} R'_{i_1 i_2}(g)\, R_{j_2 j_1}(g^{-1}) = 0}$ (*)

Now, consider two maps $G \to \mathbb{C}$: 1) $R'_{i_1 i_2} : g \mapsto R'_{i_1 i_2}(g)$

2) $\widetilde{R}_{j_2 j_1} : g \mapsto \overline{R_{j_2 j_1}(g^{-1})}$.

Using the inner product on $\mathrm{Maps}(G, \mathbb{C})$ (Prop. 6),
(*) becomes <u>$(R'_{i_1 i_2}, \widetilde{R}_{j_2 j_1}) = 0$</u> $\forall i_1, i_2, j_1, j_2$. (**)

Further, $\chi^V(g) = \mathrm{tr}(\rho_V(g)) = \mathrm{tr}(R(g)) = \sum_{j=1}^{n} R_{jj}(g) \underset{\substack{\uparrow \\ \text{since } \overline{\chi^V(g^{-1})} = \chi^V(g)}}{=} \sum_{j=1}^{n} \widetilde{R}_{j_2 j_1}(g)$,
and $\chi^W(g) = \sum_{i=1}^{n'} R'_{ii}(g)$.

So (**) $\Rightarrow$ <u>$(\chi^W, \chi^V) = 0$</u>.

Case $V=W$: $\forall \varphi,\ \varphi^{av}=\lambda \, Id_V,\ \lambda=\frac{tr\,\varphi}{\dim_{\mathbb{C}}(V)} \Rightarrow \forall F\in Mat_{n\times n}(\mathbb{C})$,

$$\frac{1}{\#G}\sum_{g\in G} R(g)FR(g)^{-1}=\lambda I.$$

$$\underset{\substack{\uparrow\\ \text{consider each cell}}}{\Rightarrow} \forall 1\le i_1,j_1\le n,\ \frac{1}{\#G}\sum_{g\in G}\sum_{1\le i_2,j_2\le n} R_{i_1i_2}(g)F_{i_2j_2}R_{j_2j_1}(g^{-1})=\frac{\sum_{i=1}^{n}F_{i,i}}{n}\delta_{i_1,j_1}$$

$$\underset{\substack{\uparrow\\ \text{consider each } F_{ij}}}{\Rightarrow} \forall 1\le i_1,j_1,i_2,j_2\le n,\ \boxed{\frac{1}{\#G}\sum_{g\in G} R_{i_1i_2}(g)R_{j_2j_1}(g^{-1})=\frac{\delta_{i_1,j_1}\delta_{i_2,j_2}}{n}}$$

$$\Rightarrow (R_{i_1i_2},\hat{R}_{j_2j_1})=\frac{1}{n}\delta_{i_1,j_1}\delta_{i_2,j_2}$$

$$\Rightarrow (R_{ii},\hat{R}_{jj})=\frac{1}{n}\delta_{ij}$$

$$\Rightarrow (\chi^V,\chi^V)=\Big(\sum_{i=1}^{n}R_{ii},\ \sum_{j=1}^{n}\hat{R}_{jj}\Big)=\frac{1}{n}\sum_{i,j=1}^{n}\delta_{i,j}=1.$$

Conclusion: χ^V, $V\in Irrep(G)$, are orthonormal.

It remains to show that they span the whole $CF(G)$.

Assume the contrary: $\exists\, \varphi\in CF(G)$ with $(\chi^V,\varphi)=0\ \forall V\in Irrep(G)$, $\varphi\neq 0$.

Because of the complete reducibility, this implies $(\chi^V,\varphi)=0\ \forall (V,\rho)\in Rep(G)$.

Consider $\rho_\varphi := \sum_{g\in G}\underbrace{\overline{\varphi(g)}}_{\in\mathbb{C}}\underbrace{\rho(g)}_{\in Aut_{\mathbb{C}}(V)}\in End_{\mathbb{C}}(V)$.

This lin. transformation has remarkable properties:

- $\rho_\varphi^{av}=\frac{1}{\#G}\sum_{h\in G}\sum_{g\in G}\overline{\varphi(g)}\rho(h)\rho(g)\rho(h)^{-1}=\frac{1}{\#G}\sum_{h,g\in G}\overline{\varphi(g)}\rho(hgh^{-1}) \underset{\substack{\uparrow\\ \varphi \text{ is a class function}}}{=}$

  $=\frac{1}{\#G}\sum_{h,g\in G}\overline{\varphi(hgh^{-1})}\rho(hgh^{-1}) \underset{\substack{\uparrow\\ \text{put } k=hgh^{-1}}}{=} \frac{1}{\#G}\sum_{h,k\in G}\overline{\varphi(k)}\rho(k)=$

  $=\rho_\varphi$

  (that is, $\rho_\varphi\in End_G(V)$);

- $tr(\rho_\varphi)=\sum_{g\in G}\overline{\varphi(g)}\,tr(\rho(g))=\sum_{g\in G}\overline{\varphi(g)}\chi^V(g)=\#G\,(\chi^V,\varphi)=0$

If V is irred., Lemma 11 gives $\rho_\varphi=\rho_\varphi^{av}=\frac{tr(\rho_\varphi)}{\dim_{\mathbb{C}}(V)}Id_V=0$.

But then $\rho_\Phi = 0$ for general reps (V, ρ), since

- the ρ_Φ construction respects direct sums: $(\rho \oplus \rho')_\Phi = \rho_\Phi \oplus \rho'_\Phi$

  { this is clearer at the level of matrices:

$$\sum_{g \in G} \overline{\Phi(g)} \begin{pmatrix} M(g) & 0 \\ 0 & M'(g) \end{pmatrix} = \begin{pmatrix} \sum_g \overline{\Phi(g)} M(g) & 0 \\ 0 & \sum_g \overline{\Phi(g)} M(g) \end{pmatrix}$$

- $V \cong V' = \bigoplus_i m_i V_i$, $(V_i, \rho_i) \in Irrep(G)$, $(V', \rho') \in Rep(G)$
- $\rho'_\Phi = \bigoplus_i m_i (\rho_i)_\Phi = 0 \Leftarrow \forall i, (\rho_i)_\Phi = 0.$
- $V \cong V'$ means $\exists\, \psi \in Iso_G(V, V')$,

  ψ G-linear $\Rightarrow \psi \rho_\Phi = \sum_g \overline{\Phi(g)}\, \psi \rho(g) = \sum_g \overline{\Phi(g)}\, \rho'(g) \psi = \rho'_\Phi \psi$

  $\Rightarrow \rho_\Phi = \psi^{-1} \rho'_\Phi \psi = 0.$

In particular, $\rho^{reg}{}_\Phi = 0$ for the regular rep. (V^{reg}, ρ^{reg}).

But then $0 = \rho^{reg}{}_\Phi (e_1) = \sum_{g \in G} \overline{\Phi(g)}\, \rho^{reg}(g)(e_1) = \sum_{g \in G} \overline{\Phi(g)}\, e_g$.

Since the e_g form a basis of V^{reg}, $\overline{\Phi(g)} = 0$ for all $g \in G$,

so $\Phi = 0$ – contradiction.

This (finally!) terminates the proof of Prop. 7.

Let us now look at a (rescaled) character table as a matrix:

$\#C$	n_1	$\dots$	n_K	
$V \backslash C$	$[g_1]$	$\dots$	$[g_K]$	} conj(G)
V_1				
$\vdots$		$\frac{\sqrt{n_j}}{\sqrt{\#G}} \chi^{V_i}(g_j)$		
V_K				

Irrep(G)

Schur's orthogonality relations:
The rows & the columns of this table form two families of orthonormal vectors.

- For the rows it is a part of Prop. 7:

$$\delta_{i,i'} = (\chi^{V_i}, \chi^{V_{i'}}) = \frac{1}{\#G} \sum_{g \in G} \chi^{V_i}(g) \overline{\chi^{V_{i'}}(g)} = \frac{1}{\#G} \sum_{j=1}^{K} n_j \chi^{V_i}(g_j) \overline{\chi^{V_{i'}}(g_j)} = \sum_{j=1}^{K} \frac{\sqrt{n_j}}{\sqrt{\#G}} \chi^{V_i}(g_j) \cdot \overline{\frac{\sqrt{n_j}}{\sqrt{\#G}} \chi^{V_{i'}}(g_j)}$$

· For the columns, it is proved in

Prop. 13: $\forall g,h\in G$, $\boxed{\sum_{V\in \mathrm{Irrep}(G)} \chi^V(g)\overline{\chi^V(h)} = \frac{\#G}{\#[h]}\mathbb{1}_{g\sim h}}$

Here: · $\sim$ is the conjugacy relation

· $\mathbb{1}_{g\sim h}=\begin{cases}1, & g\sim h\\ 0, & g\nsim h.\end{cases}$

□ Recall the indicator class functions $\mathbb{1}_{[h]}(g)=\begin{cases}1, & g\in[h],\\ 0, & g\notin[h].\end{cases}$

Prop. 7 $\Rightarrow$ $\mathbb{1}_{[h]}=\sum_{V\in\mathrm{Irrep}(G)}(\mathbb{1}_{[h]},\chi^V)\chi^V=\frac{1}{\#G}\sum_V\underbrace{\sum_{g\in G}\mathbb{1}_{[h]}(g)}_{=\mathbb{1}_{g\sim h}}\overline{\chi^V(g)}\chi^V=$

$=\frac{1}{\#G}\sum_V\sum_{g\sim h}\overline{\chi^V(g)}\chi^V \underset{\uparrow}{=} \frac{\#[h]}{\#G}\sum_V\overline{\chi^V(h)}\chi^V.$

$\chi^V(g)=\chi^V(h)$ when $g\sim h$

Evaluate both sides on $g\in G$: $\mathbb{1}_{g\sim h}=\mathbb{1}_{[h]}(g)=\frac{\#[h]}{\#G}\sum_V\overline{\chi^V(h)}\chi^V(g)$. ⊠

Now let us return to the endomorphisms p_φ from page 4. Their properties remain valid in a very general setting, with the same proofs:

Prop. 12: $\forall (V,p)\in\mathrm{Rep}(G)$, $\forall\,\varphi\in CF(G)$,

(1) $p_\varphi=p_\varphi^{av}\in\mathrm{End}_G(V)$

(2) $\mathrm{tr}(p_\varphi)=\#G\cdot(\chi^V,\varphi)$.

We'll use this result to show that, even if decomposition of reps into irreps is in general not unique (Lecture 5), there is a coarser version of this decomposition which is unique:

Prop. 14: $\mathrm{Irrep}(G)=\{V_1,\dots,V_K\}$,

$(V,p)\in\mathrm{Rep}(G)$, $\underbrace{V=\bigoplus_{i=1}^K W_i,\ \text{where } W_i\cong m_iV_i \text{ for some } m_i\in\mathbb{N}\cup\{0\}}_{(*)}$

$\Rightarrow W_i=p_{\chi^{V_i}}(V)$.

Thus $(*)$ uniquely determines the decomposition $V=\bigoplus_{i=1}^K W_i$, which is called the **canonical decomposition** of V.

The components W_i collect together isomorphic irreducible subreps of V, and are thus called the **isotypic components** of V.

$\square$ Decompose each W_i into subreps $W_{i,1}, \dots, W_{i,m_i}$, with $W_{i,j} \cong V_i$.
All the $W_{i,j}$ are G-invariant $\Rightarrow$ $\rho_{\chi^{V_\ell}} = \sum_{g \in G} \overline{\chi^{V_\ell}(g)}\, \rho(g)$ restricts to $W_{i,j}$ for all ℓ. Its restriction is $(\rho_{i,j})_{\chi^{V_\ell}}$, where $\rho_{i,j}$ is the restriction of ρ to $W_{i,j}$.

$W_{i,j} \cong V_i \Rightarrow W_{i,j}$ is irreducible, and $\chi^{W_{i,j}} = \chi^{V_i}$.

Hence $\underline{(\rho_{i,j})_{\chi^{V_\ell}}} = \dfrac{\operatorname{tr}((\rho_{i,j})_{\chi^{V_\ell}})}{\dim_{\mathbb{C}} W_{i,j}} \mathrm{Id}_{W_{i,j}} = \dfrac{\#G\,(\chi^{W_{i,j}}, \chi^{V_\ell})}{\dim_{\mathbb{C}} V_i} \mathrm{Id} = \dfrac{\#G}{\dim_{\mathbb{C}} V_i} (\chi^{V_i}, \chi^{V_\ell}) \mathrm{Id}$

$$\underline{= \delta_{i,\ell} \frac{\#G}{\dim_{\mathbb{C}} V_i} \mathrm{Id}_{W_{i,j}}}.$$

So $\rho_{\chi^{V_\ell}}$ is zero on all W_i with $i \neq \ell$, and

- multiplication by $\dfrac{\#G}{\dim_{\mathbb{C}} V_\ell} \neq 0$ on W_ℓ.

Conclusion: $\rho_{\chi^{V_\ell}}(V) = \bigoplus_i \rho_{\chi^{V_\ell}}(W_i) = W_\ell$.

$\boxtimes$