Lecture 10: Limits at infinity

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MA1S11A: Calculus with Applications for Scientists

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1 The notion of limit at infinity

We have seen that vertical asymptotes can be described mathematically using the notion of infinite limit. Today we will learn how to talk rigorously about horizontal and oblique asymptotes. For this we will need a new type of limits.

Definition. A function f is said to **have the limit L as x tends to $+\infty$** if the values $f(x)$ get as close as we like to L as x increases without bound. In this case, one writes

$$\lim_{x\to+\infty} f(x) = L, \qquad \text{or} \qquad f(x) \underset{x\to+\infty}{\longrightarrow} L.$$

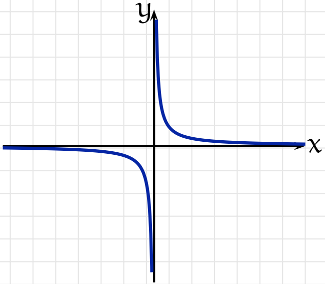
Similarly, a function f is said to **have the limit L as x tends to $-\infty$** if the values $f(x)$ get as close as we like to L as x decreases without bound. In this case, one writes

$$\lim_{x\to-\infty} f(x) = L, \qquad \text{or} \qquad f(x) \underset{x\to-\infty}{\longrightarrow} L.$$

Here expressions “tends to” and “approaches” are synonymous.

1 The notion of limit at infinity

Example. Consider the function $f(x) = 1/x$:



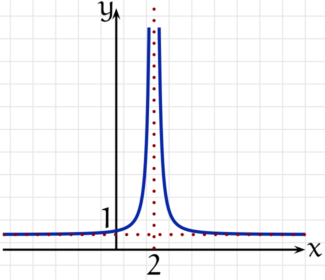
In this case, we have

$$\lim_{x\to+\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

2 Limits and asymptotes

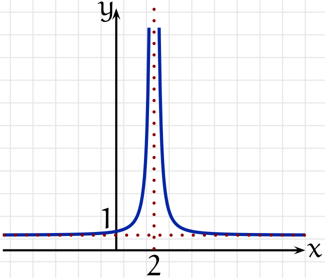
vertical asymptote $x = a$	$\lim_{x \to a^\pm} f(x) = \pm\infty$	infinite limit at a
horizontal asymptote $y = L$	$\lim_{x \to \pm\infty} f(x) = L$	finite limit at infinity

Example. Consider the function $f(x) = 1/(x-2)^2 + 1$:



2 Limits and asymptotes

Example. Consider the function $f(x) = 1/(x-2)^2 + 1$:



In this case, we have

$$\lim_{x\to+\infty} f(x) = 1 \text{ and } \lim_{x\to-\infty} f(x) = 1,$$

and $y = 1$ is the only horizontal asymptote here.

Also, the limit is finite (computed by evaluation) at all points except 2, where $\lim_{x\to 2} f(x) = +\infty$. So, $x = 2$ is the only vertical asymptote.

3 Infinite limits at infinity

Definition. A function f is said to **have the limit $+\infty$ as x tends to $+\infty$ (or $-\infty$)** if the values $f(x)$ increase without bound as x increases without bound (decreases without bound). In this case, one writes

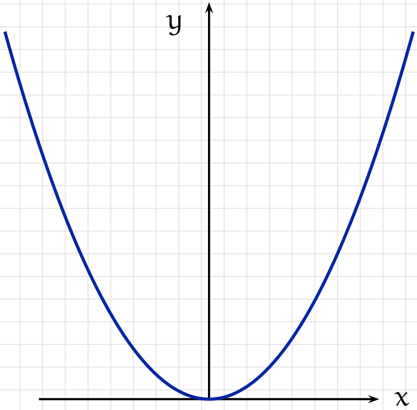
$$\lim_{x\to\pm\infty} f(x) = +\infty, \qquad \text{or} \qquad f(x) \underset{x\to\pm\infty}{\longrightarrow} +\infty.$$

Similarly, a function f is said to **have the limit $-\infty$ as x tends to $+\infty$ (or $-\infty$)** if the values $f(x)$ decrease without bound as x increases without bound (decreases without bound). In this case, one writes

$$\lim_{x\to\pm\infty} f(x) = -\infty, \qquad \text{or} \qquad f(x) \underset{x\to\pm\infty}{\longrightarrow} -\infty.$$

3 Infinite limits at infinity

Example. Consider the function x^2:

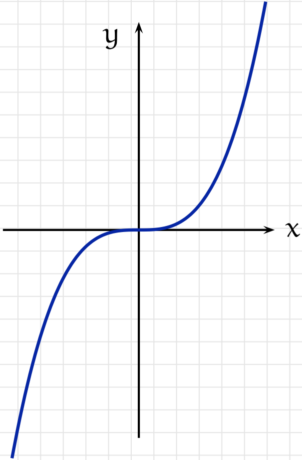


In this case, we have

$$\lim_{x\to+\infty} f(x) = +\infty \text{ and } \lim_{x\to-\infty} f(x) = +\infty.$$

3 Infinite limits at infinity

Example. Consider the function x^3:



In this case, we have

$$\lim_{x \to +\infty} f(x) = +\infty \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

4 Limits of polynomial functions at infinity

We have already seen that for large $|x|$, a polynomial

$$f(x) = a_n x^n + \cdots + a_1 x + a_0, \quad \text{where } a_n \neq 0,$$

behaves as its leading term $a_n x^n$. This makes the following result very natural:

Theorem. Suppose that $f(x) = a_n x^n + \cdots + a_1 x + a_0$, and $a_n \neq 0$. Then

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (a_n x^n),$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (a_n x^n).$$

If $n > 0$, then these limits are infinite, $\pm\infty$. The signs depend on the sign of a_n and on the parity of n (whether it is even or odd).

Informally, this theorem says that the limiting behaviour at infinity of a polynomial exactly matches the behaviour of its highest degree term.

4 Limits of polynomial functions at infinity

Theorem. Suppose that $f(x) = a_n x^n + \cdots + a_1 x + a_0$, and $a_n \neq 0$. Then

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (a_n x^n),$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (a_n x^n).$$

Proof. We have

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right). \end{aligned}$$

The expression in the brackets clearly has the limit 1 as $x \to \pm\infty$, since all terms except for 1 have the limit 0. ☐

Example.

$$\lim_{x \to +\infty} (1 - x)(x + 2)^2 = \lim_{x \to +\infty} -x^3 = -\infty;$$

$$\lim_{x \to -\infty} (1 - x)(x + 2)^2 = \lim_{x \to -\infty} -x^3 = +\infty.$$

5 Limits of rational functions at infinity

Theorem. Suppose that $f(x) = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$, with $a_n \neq 0$, $b_m \neq 0$. Then

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{a_n x^n}{b_m x^m},$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{a_n x^n}{b_m x^m}.$$

Proof. We have

$$f(x) = \frac{a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)}{b_m x^m \left(1 + \frac{b_{m-1}}{b_m x} + \cdots + \frac{b_1}{b_m x^{m-1}} + \frac{b_0}{b_m x^m}\right)}$$
$$= \frac{a_n x^n}{b_m x^m} \frac{1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}}{1 + \frac{b_{m-1}}{b_m x} + \cdots + \frac{b_1}{b_m x^{m-1}} + \frac{b_0}{b_m x^m}}.$$

Both the numerator and the denominator of the second fraction tend to 1 as $x \to \pm\infty$, since all terms except for 1 have the limit 0. □

5 Limits of rational functions at infinity

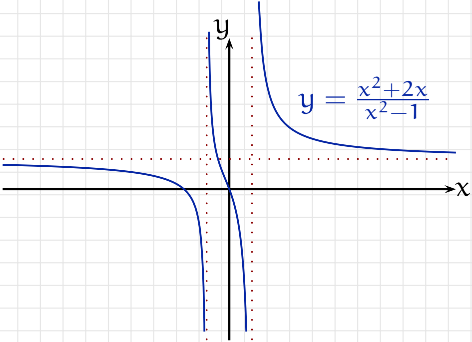
Examples.

$$\lim_{x\to+\infty} \frac{3x+5}{6x-8} = \lim_{x\to+\infty} \frac{3x}{6x} \cdot \frac{1+\frac{5}{3x}}{1-\frac{8}{6x}} = \lim_{x\to+\infty} \frac{3x}{6x} = \frac{1}{2}.$$

$$\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5} = \lim_{x\to-\infty} \frac{4x^2}{2x^3} = \lim_{x\to-\infty} \frac{2}{x} = 0.$$

$$\lim_{x\to+\infty} \frac{5x^3-2x^2+1}{1-3x} = \lim_{x\to+\infty} \frac{5x^3}{-3x} = \lim_{x\to+\infty} -\frac{5}{3}x^2 = -\infty.$$

5 Limits of rational functions at infinity

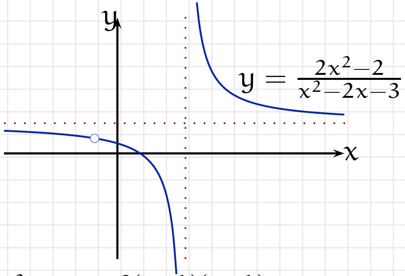


The graph of $f(x) = \frac{x^2+2x}{x^2-1} = \frac{x(x+2)}{(x-1)(x+1)}$ has vertical asymptotes $x = 1$ and $x = -1$, and a horizontal asymptote $y = 1$. Now we know which limits control those asymptotes:

$$\lim_{x \to -1^-} f(x) = -\infty, \lim_{x \to -1^+} f(x) = +\infty, \lim_{x \to 1^-} f(x) = -\infty, \lim_{x \to 1^+} f(x) = +\infty$$

for the vertical asymptotes, and $\lim_{x \to \pm\infty} \frac{x^2+2x}{x^2-1} = \lim_{x \to \pm\infty} \frac{x^2}{x^2} = 1$ for the horizontal asymptote.

5 Limits of rational functions at infinity



The graph of $f(x) = \frac{2x^2-2}{x^2-2x-3} = \frac{2(x-1)(x+1)}{(x+1)(x-3)}$ has a vertical asymptote $x = 3$, a horizontal asymptote $y = 2$, and also the "cut" $(-1, 1)$ which it approaches both on the left and on the right but does not touch. This is controlled by the limits

$$\lim_{x\to 3^-} f(x) = -\infty, \lim_{x\to 3^+} f(x) = +\infty,$$

$$\lim_{x\to -\infty} f(x) = 2, \lim_{x\to +\infty} f(x) = 2,$$

$$\lim_{x\to -1} f(x) = 1.$$

6 Limits of algebraic functions at infinity

As usual, there are no general recipes for this wide class of functions. We will just have a look at several examples.

Examples.

For ratios involving square roots, the leading term method still works:

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to+\infty} \frac{\sqrt{x^2(1+\frac{2}{x^2})}}{3x(1-\frac{2}{x})} = \lim_{x\to+\infty} \frac{|x|}{3x} = \frac{1}{3}.$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to-\infty} \frac{\sqrt{x^2(1+\frac{2}{x^2})}}{3x(1-\frac{2}{x})} = \lim_{x\to-\infty} \frac{|x|}{3x} = -\frac{1}{3}.$$

This is our first example of function with two horizontal asymptotes: $y = \frac{1}{3}$ and $y = -\frac{1}{3}$.

6 Limits of algebraic functions at infinity

For differences $\sqrt{f(x)} - \sqrt{g(x)}$, or simply $\sqrt{f(x)} - g(x)$, where both f and g tend to $+\infty$ at the point of interest (finite or infinite), the formula $a - b = \frac{a^2-b^2}{a+b}$ can be useful.

$$\lim_{x\to+\infty}\left(\sqrt{x^6+5} - x^3\right) = \lim_{x\to+\infty}\frac{(\sqrt{x^6+5})^2 - (x^3)^2}{\sqrt{x^6+5}+x^3} = \lim_{x\to+\infty}\frac{5}{\sqrt{x^6+5}+x^3} = 0.$$

$$\lim_{x\to+\infty}\left(\sqrt{x^6+5x^3} - x^3\right) = \lim_{x\to+\infty}\frac{(\sqrt{x^6+5x^3})^2 - (x^3)^2}{\sqrt{x^6+5x^3}+x^3} = \lim_{x\to+\infty}\frac{5x^3}{\sqrt{x^6+5x^3}+x^3} = \lim_{x\to+\infty}\frac{5x^3}{x^3\left(\sqrt{1+\frac{5}{x^3}}+1\right)} = \frac{5}{2}.$$

We used that for all $x > 0$, all the expressions under the square roots are positive, and division by zero never occurs.

7 Limits of trigonometric functions at infinity

Since $\sin x$ and $\cos x$ oscillate between -1 and 1 as $x \to \pm\infty$, neither of these functions has a limit at infinity.

However, limits like $\lim_{x\to+\infty} \frac{\sin x}{x}$ might exist. Indeed, as $x \to +\infty$, the value of $\sin x$ is between -1 and 1, and the value of x increases without bound, so the ratio of these quantities has the limit 0:

$$\lim_{x\to+\infty} \frac{\sin x}{x} = 0.$$

⚠ Please do **not** do anything like that:

$$\lim_{x\to+\infty} \frac{\sin x}{x} = \lim_{x\to+\infty} \frac{\sin \cancel{x}}{\cancel{x}} = \lim_{x\to+\infty} \frac{\sin}{1} = \sin.$$