1. The Philosophy of Representation Theory

Two viewpoints:

① abstract algebra $\xrightarrow{\text{representations}}$ linear algebra

groups, rings, Lie algebras, etc. $\underset{\text{examples}}{\rightleftarrows}$ matrices

$\boxed{\text{Idea}}$: Replace complicated algebraic structures by easier ones (here matrices).

② How can one describe a group $\mathcal{G}$ (or a ring or a Lie algebra)?

Our favourite example for this lecture: symmetric groups S_n.

(a) elements & operations

satisfying $\cdot$, $1 \in \mathcal{G}$, $(\ )^{-1}$

the product symbol is omitted in what follows

$\rightarrow \forall g_i \in \mathcal{G}$, $g_1(g_2g_3) = (g_1g_2)g_3$

$\rightarrow \forall g \in \mathcal{G}$, $1g = g1 = g$

$\rightarrow \forall g \in \mathcal{G}$, $gg^{-1} = g^{-1}g = 1$.

Pb: Given a product (e.g., by a multiplication table), it can be tedious to verify its associativity.

permutations $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & & \sigma(n) \end{pmatrix}$

$\cdot$ is the composition: $\sigma_1\sigma_2 : i \mapsto \sigma_1(\sigma_2(i))$

1 is the trivial perm. $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$

σ^{-1} is the inverse perm.: $\sigma(i) \mapsto i$

(b) generators & relations

$\mathcal{G} = \langle g_1, g_2, \dots \mid l_1 = r_1, l_2 = r_2, \dots \rangle$

where the l_i and the r_i are words in the generators g_j & their inverses g_j^{-1},

e.g. $\mathbb{Z}_n = \langle g \mid g^n = 1 \rangle$,

$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = 1, g_1g_2 = g_2g_1 \rangle$

$$S_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i^2 = 1, \text{ for } 1 \le i \le n-1, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ for } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \qquad \text{for } 1 \le i \le n-2 \end{array} \right\rangle$$

$$\sigma_i = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & i & i+2 & \cdots & n \end{pmatrix}$$

In the cycle notation, $\sigma_i = (i\ \ i+1)$.

Pb: Elements of G, written as words in $g_j^{\pm 1}$, are difficult to compare modulo the relations $l_i = r_i$.

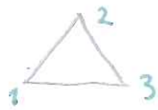
Thm: These generators & relations describe the group of permutations from (a).

(c) as a group of symmetries

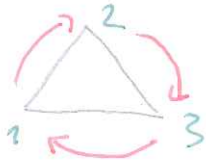
Idea: To understand a group, let it act on something.

S_n = the group of symmetries (GoS) of the n-point set

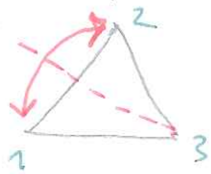
S_3 = the GoS of an equilateral triangle



Ex: (123) = rotation:



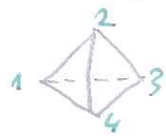
(12) = reflection



Alternatively,

S_3 = the GoS of the plane $\mathbb{R}^2$ fixing an equilateral triangle

S_4 = the GoS of a regular tetrahedron



A_3 = the group of orientation-preserving symmetries of Δ.

alternating group

Ex: $(123) \in A_3$, $(12) \notin A_3$

Compare: The litmus paper test in chemistry:

① replace smth complicated (a liquid) by smth simple (the red-blue colour scale);

② get insights on smth complicated (a liquid) by letting it act on smth (the litmus paper).

Def.: A representation of a group G is a finite-dimensional space V over $\mathbb{C}$ equipped with:

Version 1: (cf. viewpoint ①) — more concise

a group morphism $\boxed{\rho : G \xrightarrow[\text{grp}]{} \mathrm{Aut}_{\mathbb{C}}(V)}$

$\mathbb{C}$-linear automorphisms of V

$\mathrm{Aut}_{\mathbb{C}}(V) = GL(V, \mathbb{C}) = \mathrm{Mat}^{*}_{n\times n}(\mathbb{C})$ ← when a basis of V is fixed (invertible matrices)

i.e., $\forall g, g' \in G$, $\rho(gg') = \rho(g)\rho(g')$

Exo (exercise): $\Rightarrow$
- $\rho(1) = \mathrm{Id}_V$
- $\forall g \in G$, $\rho(g^{-1}) = (\rho(g))^{-1}$

Version 2: (cf. viewpoint ②) — more intuitive

a map $\boxed{\begin{array}{l} G \times V \to V \\ (g, v) \mapsto g\cdot v \end{array}}$ satisfying:

$\forall g, g' \in G$, $\forall \lambda \in \mathbb{C}$, $\forall v, v' \in V$,

(i) $g\cdot(\lambda v + v') = \lambda\, g\cdot v + g\cdot v'$

(ii) $(gg')\cdot v = g\cdot(g'\cdot v)$

(iii) $1\cdot v = v$

i.e., $\cdot$ is a linear G-action

Exo: The two versions of this definition are equivalent.

Rmk: One can also consider infinite-dimensional reps over any field.

Terminology:
- One refers to a representation as (V, ρ), or simply V, or simply ρ, when the remaining data is clear from the context.
- A rep. of G is also called G-representation, or G-module.
- V is called the representation space of ρ.
- $\deg \rho := \dim_{\mathbb{C}} V$

The aim of representation theory is to understand

$\mathrm{Rep}(G)$ = the set of reps of G.
+ additional structure

Examples of representations:

(0) zero rep.: $V=\{0\}$

(1) trivial rep.: $V=\mathbb{C}$, $\rho(g)=1\in Mat_{1\times1}(\mathbb{C})$ for all $g\in G$.
i.e., $g\cdot v=v$ for all $g\in G$, $v\in V$

(2) Let X be a finite G-set $=$ set with a G-action, i.e., a map $G\times X\to X$ satisfying (ii) & (iii).
Denote by $\mathbb{C}X$ the vector space over $\mathbb{C}$ with the basis e_x, $x\in X$. Then $\mathbb{C}X$ is a G-rep. of degree $\#X$, via $g\cdot \sum_{x\in X} \underset{\mathbb{C}}{\underset{\rotatebox{90}{$\in$}}{\alpha_x}} e_x = \sum_{x\in X}\alpha_x e_{g\cdot x}$. It is called a permutation rep.

Ex.: For $X_n:=\{1,2,\dots,n\}$, $\mathbb{C}X_n$ is an S_n-rep. of degree n, with $\sigma\cdot e_i=e_{\sigma(i)}$.

(3) A group G acts on itself by $g\cdot g'=gg'$. Indeed,
(ii) $(g_1g_2)\cdot g_3=(g_1g_2)g_3=g_1(g_2g_3)=g_1\cdot(g_2\cdot g_3)$,
(iii) $1\cdot g=1g=g$.
Then, according to (2), $\mathbb{C}G$ is a G-rep. of degree $\#G$, via $g\cdot e_{g'}=e_{gg'}$. It is the left regular rep. of G.

Ex.: $\mathbb{C}S_n$ is an S_n-rep. of degree $n!$, with $\sigma\cdot e_{\sigma'}=e_{\sigma\sigma'}$.

We will learn how to describe all the reps of a given group.

Applications of representation theory

→ understanding the structure of groups
  ex.: classification of finite simple groups

→ Galois theory

  ex.: the non-solvability of degree 5 polynomial equations follows from certain properties of the group S_5 (acting on the roots of a polynomial), established using the study of $Rep(S_5)$

→ study of quantities invariant under certain symmetries in physics

→ study of symmetries of crystal lattices in crystallography

→ modular forms in number theory

→ Fourier analysis

→ invariants of braids & knots constructed from reps of braid groups

→ graph theory

  ex.: reps of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ are used to study
  $g_{n,k} = \#\{\text{graphs with } n \text{ vertices } \& \ k \text{ edges}\}/\text{iso}$

→ probability theory

  ex.: the famous Bayer-Diaconis 1992 theorem "A deck of 54 cards should be shuffled 7 times" uses $Rep(S_{54})$.