Homework/Tutorial 9

A complete solution to question 1 is worth 6 marks (1 for each integral and 2 for the last one); for questions 2 to 3 it is 1 mark; for question 4 it is 2 marks.

What this homework is about

You will practice in integrating functions and applying integral and differential calculi in various contexts.

Reminder

A function F is called an **antiderivative**, or an **indefinite integral**, of a function f if $F' = f$. Notation: $\int f(x)\,dx = F(x) + C$, where C is an arbitrary constant. If f is defined on a disjoint union of intervals or rays, then a separate constant should be taken on each of them.

Integration rules:

- linearity: for any real numbers $c_1, \ldots, c_n$, we have
$$\int (c_1 f_1(x) + \cdots + c_n f_n(x))\,dx = c_1 \int f_1(x)\,dx + \cdots + c_n \int f_n(x)\,dx;$$
- u-substitution:
$$F' = f \qquad \Longrightarrow \qquad \int f(g(x))g'(x)\,dx = \int f(u)\,du = F(g(x)) + C;$$
- integration by parts:
$$F' = f \text{ and } G' = g \qquad \Longrightarrow \qquad \int [f(x)G(x)]\,dx = F(x)G(x) - \int [F(x)g(x)]\,dx$$
(briefly, $\int G\,dF = FG - \int F\,dG$).

Applications of differentiation:

- For **rectilinear motion** (i.e., motion along a line), its coordinate $x(t)$, velocity $v(t)$, and acceleration $a(t)$ (where t is the time variable) are related as follows:
$$v(t) = x'(t), \qquad a(t) = v'(t) = x''(t).$$
- **Algorithm for finding global extrema of a function f on** $[a, b]$:
  1) Determine all **critical points** of f (i.e., c for which $f'(c)$ does not exist or $f'(c) = 0$).
  2) Exclude **stationary points** ($f'(c) = 0$) failing the first or the second derivative tests.
  3) Evaluate f at the remaining critical points and at the endpoints a and b.
  4) Among the values obtained, choose the minimal and the maximal ones.
- **Newton's method** for approximating a zero of a differentiable function f: Starting with an initial estimate c_0, compute further estimates $c_1, c_2, \ldots$ by the recursive formula
$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)}.$$
In many cases, these estimates will approach a solution of $f(x) = 0$, and do it very fast.
- **L'Hôpital's rule:** Assume that the functions f and g satisfy:
  1) $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ or $\pm\infty$;
  2) f and g are differentiable on (a, b), except possibly at c;
  3) $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$;
  4) $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists.

  Then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$.

Questions

1. Compute the following indefinite integrals:

   (a) $\int (x+1)^2 3^x \, dx$; (b) $\int \frac{e^x \, dx}{1+e^{2x}}$; (c) $\int \frac{x-1}{x^2} \, dx$;

   (d) $\int \sin(2x) e^{\cos(x)^2} \, dx$ (*hint*: you might need some trigonometric formulas);

   (e) $\int \sqrt{1-x^2} \, dx$ (*hint*: at some point you might use the formula
   $$\frac{x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2},$$
   which you will need to verify).

2. Compute $\lim_{x \to 1} \frac{2^x - 2}{\log_2(x)}$.

3. Write down the recursive formula of Newton's method for the function $f(x) = x^2 - a$. Here $a > 0$ is a fixed real number. Briefly explain how it can be used to approximate $\sqrt{a}$ with the help of a device that can only add, multiply, and divide real numbers.

4. A fly moves back and forth along a straight horizontal rod. Its instantaneous velocity v (in m/s) between the time $t = 0$ and $t = 4$ (in s) is given by the formula
   $$v(t) = \frac{t}{4} \sin \frac{\pi t}{2},$$
   and you know that it first moves to the right. At what time will it be at the leftmost and at the rightmost points of its trajectory? (*Hint:* You might need to compute the antiderivative of v, and to look at the lecture notes on rectilinear motion (Lecture 19).)