Homework/Tutorial 8

What this homework is about

You will practice in analysing functions with the help of differential calculus.

Reminder

Algorithm for graphing a rational function \( f(x) = P(x)/Q(x) \)

1. Check if \( f \) is given in the reduced form (i.e., the polynomials \( P \) and \( Q \) have no common factors). If not, find its reduced form.
2. Determine if the graph has symmetries about the \( y \)-axis / the origin, i.e., whether \( f \) is even / odd.
3. Find where and how the graph meets the \( x \)-axis, i.e., compute the roots of \( f \) and their multiplicities. (A root of \( f \) is a root \( c \) of \( P \). It is of multiplicity \( m \) if \((x-c)^m \) divides \( P(x) \) but \((x-c)^{m+1} \) does not.)
4. Find where the graph meets the \( y \)-axis, i.e., compute \( f(0) \).
5. Determine all vertical asymptotes and check if there is a sign change across them, i.e., compute the poles of \( f \) and their multiplicities. (A pole of \( f \) is a root of \( Q \).)
6. Describe the behaviour of \( f \) at \( \pm \infty \): compute \( \lim_{x \to \pm \infty} f(x) \), and find the curvilinear asymptote of the graph. (For this you need to divide \( P \) by \( Q \), and use this to present \( f \) as \( S(x) + R(x)/Q(x) \) with \( \deg R < \deg Q \).)
7. Find the sign of \( f \) on each interval between the \( x \)-intercepts and the vertical asymptotes.
8. Determine where \( f \) is increasing/decreasing, and find all critical points, and local and global extrema. For this, analyse the sign of \( f' \) (if it exists).
9. Determine where \( f \) is concave up/down, and find all inflection points. For this, analyse the sign of \( f'' \) (if it exists).
10. Sketch the graph of \( f \).

Question

Analyse the following rational function using the plan above, and sketch its graph:

\[
 f(x) = \frac{x^4 - 2x^3 + x^2}{x^2 - 2x}.
\]

Solution.

1. The rational function \( f \) is not given in its reduced form, since \( x \) divides both the numerator and the denominator. Dividing them both by \( x \), we get

\[
 f(x) = \frac{x^3 - 2x^2 + x}{x - 2}, \quad x \neq 0.
\]

This is a reduced form. Indeed, \( x - 2 \) is not a factor of \( x^3 - 2x^2 + x \), since \( x = 2 \) is not a root of \( x^3 - 2x^2 + x \) (as \( 2^3 - 2 \cdot 2^2 + 2 = 2 \neq 0 \)). From now on, we will use the notation

\[
 P(x) = x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x - 1)^2, \quad Q(x) = x - 2.
\]

2. Our function is neither odd nor even: for instance, \( f(1) = 0 \) but \( f(-1) \neq 0 \). So, its graph has no symmetries.
3. The polynomial \( P(x) = x(x - 1)^2 \) has two roots:
   (a) \( x = 0 \) (of multiplicity 1);
   (b) \( x = 1 \) (of multiplicity 2).
So, the graph of \( f \) intersects the \( x \)-axis twice:
   (a) at \( x = 0 \), where the graph changes sign and is not tangent to the \( x \)-axis;
   (b) at \( x = 1 \), where the graph keeps its sign and is tangent to the \( x \)-axis.
4. Since \( x = 0 \) is not in the natural domain of \( f \), the graph does not intersects the \( y \)-axis.
   However, from the reduced form of \( f \) we see that \( f \) has a removable singularity at 0:
   \[
   \lim_{x \to 0^+} f(x) = 0.
   \]
   So, the graph has a cut at \((0,0)\).
5. The polynomial \( Q(x) = x - 2 \) has only one simple root \( x = 2 \), which becomes a pole of \( f \)
of multiplicity 1. So, \( f \) has a vertical asymptote at \( x = 2 \), where it changes sign.
6. The polynomial long division of \( P \) by \( Q \) yields
   \[
   x^3 - 2x^2 + x = (x - 2)(x^2 + 1) + 2,
   \]
   So,
   \[
   f(x) = \frac{x^3 - 2x^2 + x}{x - 2} = \frac{(x - 2)(x^2 + 1) + 2}{x - 2} = x^2 + 1 + \frac{2}{x - 2}, \quad x \neq 0.
   \]
   This yields \( \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} (x^2 + 1) = +\infty \).
   Alternatively, you can look at the highest terms of the numerator and the denominator,
as it was done in lectures: \( \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{x^3}{x} = \lim_{x \to \pm \infty} x^2 = +\infty \).
   From the presentation \( f(x) = x^2 + 1 + \frac{2}{x - 2} \), we also see that the graph of \( f \) has a parabolic
   asymptote given by the quadratic function \( g(x) = x^2 + 1 \).
7. From \( \lim_{x \to +\infty} f(x) = +\infty \) we see that \( f \) is positive for very large \( x \). We have also established
   that \( f \) changes sign at 0 and 2 (but not at the root 1). This is summarised in the following table:

<table>
<thead>
<tr>
<th>interval</th>
<th>(−∞, 0)</th>
<th>(0, 2)</th>
<th>(2, +∞)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of ( f )</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

8. For computing \( f' \), it is convenient to use the “principal part and remainder” presentation
   of \( f \):
   \[
   f'(x) = \left( x^2 + 1 + \frac{2}{x - 2} \right)' = 2x - \frac{2}{(x - 2)^2} = 2\frac{x^3 - 4x^2 + 4x - 1}{(x - 2)^2}.
   \]
   Now, \( x = 1 \) is a root of \( x^3 - 4x^2 + 4x - 1 \) (you can guess it, or use the following fact if
   you know it: if \( c \) is a root of a polynomial \( P \) of multiplicity \( m > 1 \), then it is also a root
   of \( P' \) of multiplicity \( m - 1 \).) So,
   \[
   x^3 - 4x^2 + 4x - 1 = (x - 1)(x^2 - 3x + 1) = (x - 1)(x - \frac{3 - \sqrt{5}}{2})(x - \frac{3 + \sqrt{5}}{2}).
   \]
   Therefore, \( f' \) has three roots: 1, \( \frac{3 - \sqrt{5}}{2} \approx 0.38 \), and \( \frac{3 + \sqrt{5}}{2} \approx 2.62 \). Since they are all
   simple, \( f' \) changes sign at all of them. The denominator does not contribute to the sign:
   \( (x - 2)^2 \geq 0 \) for all \( x \). Finally, \( f'(x) > 0 \) for large \( x \). This information is sufficient for
   constructing the sign table of \( f' \):

<table>
<thead>
<tr>
<th>interval</th>
<th>((−\infty, \frac{3 - \sqrt{5}}{2}))</th>
<th>((\frac{3 - \sqrt{5}}{2}, 1))</th>
<th>(1, 2)</th>
<th>((2, \frac{3 + \sqrt{5}}{2}))</th>
<th>((\frac{3 + \sqrt{5}}{2}, +\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of ( f' )</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>sign of ( f )</td>
<td>(\downarrow)</td>
<td>(\nearrow)</td>
<td>(\downarrow)</td>
<td>(\searrow)</td>
<td>(\nearrow)</td>
</tr>
</tbody>
</table>
We included the point \(x = 2\) as a “separating point” since \(f\) has an infinite discontinuity there, and excluded \(x = 0\) since this discontinuity is removable. You can include \(x = 0\) if you prefer.

Feeding information from the table to the first derivative test, we see that \(f\) has local extrema at all its stationary points: a maximum at \(x = 1\), and minima at \(x = 3 \pm \sqrt{5}/2\). There are no other critical points, since \(f\) is differentiable on its domain \((-\infty, 0) \cup (0, 2) \cup (2, +\infty)\). Our function has no global extrema: since \(\lim_{x \to 2^-} f(x) = -\infty\) and \(\lim_{x \to 2^+} f(x) = +\infty\), \(f\) takes arbitrarily large and arbitrary small values close to \(x = 2\).

9. 

\[
f''(x) = \left(x^2 + 1 + \frac{2}{x-2}\right)'' = \frac{2x - \frac{2}{(x-2)^2}}{2} = 2\frac{4}{(x-2)^3} = 2\frac{(x - 2)^3 + 2}{(x-2)^3}.
\]

The equation \((x - 2)^3 + 2 = 0\) is equivalent to \(x - 2 = -\sqrt[3]{2}\), and has only one real solution: \(x = 2 - \sqrt[3]{2} \approx 0.74\). So, \(f''\) has only one root, \(x = 2 - \sqrt[3]{2} \approx 0.74\). Since this root is simple, \(f''\) changes sign at it. Also, \(f''\) changes sign at the discontinuity point \(x = 2\). At the removable discontinuity point \(x = 0\) nothing happens. For large \(x\), \(f''(x)\) is clearly positive. This information is sufficient for constructing the sign table of \(f''\):

<table>
<thead>
<tr>
<th>interval</th>
<th>((-\infty, 2 - \sqrt[3]{2}))</th>
<th>((2 - \sqrt[3]{2}, 2))</th>
<th>((2, +\infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of (f'')</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>concavity of (f)</td>
<td>up</td>
<td>down</td>
<td>up</td>
</tr>
</tbody>
</table>

From the table, we see that \(f\) has only one inflection point, \(x = 2 - \sqrt[3]{2}\).

10. Summarising all the available information, we can sketch the graph of \(f\):

On this graph you see

- the vertical asymptote \(x = 2\);
- parabolic shape for large and small \(x\);
- the cut at \((0, 0)\);
- the root (and local maximum) at \(x = 1\) (in blue);
- the local minima at \(x = \frac{3 \pm \sqrt{5}}{2}\) (in red);
- the inflection point at \(x = 2 - \sqrt[3]{2}\) (in violet).