Homework/Tutorial 8

What this homework is about

You will practice in analysing functions with the help of differential calculus.

Reminder

Algorithm for graphing a rational function $f(x) = P(x)/Q(x)$

1. Check if f is given in the reduced form (i.e., the polynomials P and Q have no common factors). If not, find its reduced form.
2. Determine if the graph has symmetries about the y-axis / the origin, i.e., whether f is even / odd.
3. Find where and how the graph meets the x-axis, i.e., compute the roots of f and their multiplicities. (A **root** of f is a root c of P. It is of **multiplicity** m if $(x-c)^m$ divides $P(x)$ but $(x-c)^{m+1}$ does not.)
4. Find where the graph meets the y-axis, i.e., compute $f(0)$.
5. Determine all vertical asymptotes and check if there is a sign change across them, i.e., compute the poles of f and their multiplicities. (A **pole** of f is a root of Q.)
6. Describe the behaviour of f at $\pm\infty$: compute $\lim\limits_{x\to\pm\infty} f(x)$, and find the curvilinear asymptote of the graph. (For this you need to divide P by Q, and use this to present f as $S(x) + \frac{R(x)}{Q(x)}$ with $\deg R < \deg Q$.)
7. Find the sign of f on each interval between the x-intercepts and the vertical asymptotes.
8. Determine where f is increasing/decreasing, and find all critical points, and local and global extrema. For this, analyse the sign of f' (if it exists).
9. Determine where f is concave up/down, and find all inflection points. For this, analyse the sign of f'' (if it exists).
10. Sketch the graph of f.

Question

Analyse the following rational function using the plan above, and sketch its graph:

$$f(x) = \frac{x^4 - 2x^3 + x^2}{x^2 - 2x}.$$

Solution.

1. The rational function f is not given in its reduced form, since x divides both the numerator and the denominator. Dividing them both by x, we get
$$f(x) = \frac{x^3 - 2x^2 + x}{x - 2}, \qquad x \neq 0.$$
This is a reduced form. Indeed, $x-2$ is not a factor of $x^3 - 2x^2 + x$, since $x = 2$ is not a root of $x^3 - 2x^2 + x$ (as $2^3 - 2 \cdot 2^2 + 2 = 2 \neq 0$). From now on, we will use the notation
$$P(x) = x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2, \qquad Q(x) = x - 2.$$
2. Our function is neither odd nor even: for instance, $f(1) = 0$ but $f(-1) \neq 0$. So, its graph has no symmetries.

3. The polynomial $P(x) = x(x-1)^2$ has two roots:
   (a) $x = 0$ (of multiplicity 1);
   (b) $x = 1$ (of multiplicity 2).

   So, the graph of f intersects the x-axis twice:
   (a) at $x = 0$, where the graph changes sign and is not tangent to the x-axis;
   (b) at $x = 1$, where the graph keeps its sign and is tangent to the x-axis.
4. Since $x = 0$ is not in the natural domain of f, the graph does not intersects the y-axis. However, from the reduced form of f we see that f has a removable singularity at 0:
$$\lim_{x\to 0^\pm} f(x) = 0.$$
   So, the graph has a cut at $(0, 0)$.
5. The polynomial $Q(x) = x - 2$ has only one simple root $x = 2$, which becomes a pole of f of multiplicity 1. So, f has a vertical asymptote at $x = 2$, where it changes sign.
6. The polynomial long division of P by Q yields
$$x^3 - 2x^2 + x = (x-2)(x^2+1) + 2,$$
   So,
$$f(x) = \frac{x^3 - 2x^2 + x}{x-2} = \frac{(x-2)(x^2+1)+2}{x-2} = x^2 + 1 + \frac{2}{x-2}, \qquad x \neq 0.$$
   This yields $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} (x^2+1) = +\infty$.
   Alternatively, you can look at the highest terms of the numerator and the denominator, as it was done in lectures: $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{x^3}{x} = \lim\limits_{x\to\pm\infty} x^2 = +\infty$.
   From the presentation $f(x) = x^2+1+\frac{2}{x-2}$, we also see that the graph of f has a parabolic asymptote given by the quadratic function $g(x) = x^2 + 1$.
7. From $\lim\limits_{x\to+\infty} f(x) = +\infty$ we see that f is positive for very large x. We have also established that f changes sign at 0 and 2 (but not at the root 1). This is summarised in the following table:

interval	$(-\infty, 0)$	$(0, 2)$	$(2, +\infty)$
sign of f	$+$	$-$	$+$

8. For computing f', it is convenient to use the "principal part and remainder" presentation of f:
$$f'(x) = \left(x^2 + 1 + \frac{2}{x-2}\right)' = 2x - \frac{2}{(x-2)^2} = 2\frac{x(x-2)^2 - 1}{(x-2)^2} = 2\frac{x^3 - 4x^2 + 4x - 1}{(x-2)^2}.$$
   Now, $x = 1$ is a root of $x^3 - 4x^2 + 4x - 1$ (you can guess it, or use the following fact if you know it: if c is a root of a polynomial P of multiplicity $m > 1$, then it is also a root of P' of multiplicity $m - 1$.) So,
$$x^3 - 4x^2 + 4x - 1 = (x-1)(x^2 - 3x + 1) = (x-1)(x - \frac{3+\sqrt{5}}{2})(x - \frac{3-\sqrt{5}}{2}).$$
   Therefore, f' has three roots: 1, $\frac{3-\sqrt{5}}{2} \approx 0.38$, and $\frac{3+\sqrt{5}}{2} \approx 2.62$. Since they are all simple, f' changes sign at all of them. The denominator does not contribute to the sign: $(x-2)^2 \geq 0$ for all x. Finally, $f'(x) > 0$ for large x. This information is sufficient for constructing the sign table of f':

interval	$(-\infty, \frac{3-\sqrt{5}}{2})$	$(\frac{3-\sqrt{5}}{2}, 1)$	$(1, 2)$	$(2, \frac{3+\sqrt{5}}{2})$	$(\frac{3+\sqrt{5}}{2}, +\infty)$
sign of f'	$-$	$+$	$-$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

We included the point $x = 2$ as a "separating point" since f has an infinite discontinuity there, and excluded $x = 0$ since this discontinuity is removable. You can include $x = 0$ if you prefer.
Feeding information from the table to the first derivative test, we see that f has local extrema at all its stationary points: a maximum at $x = 1$, and minima at $x = \frac{3\pm\sqrt{5}}{2}$. There are no other critical points, since f is differentiable on its domain $(-\infty, 0)\cup(0, 2)\cup(2, +\infty)$.
Our function has no global extrema: since $\lim\limits_{x\to 2^-} f(x) = -\infty$ and $\lim\limits_{x\to 2^+} f(x) = +\infty$, f takes arbitrarily large and arbitrary small values close to $x = 2$.

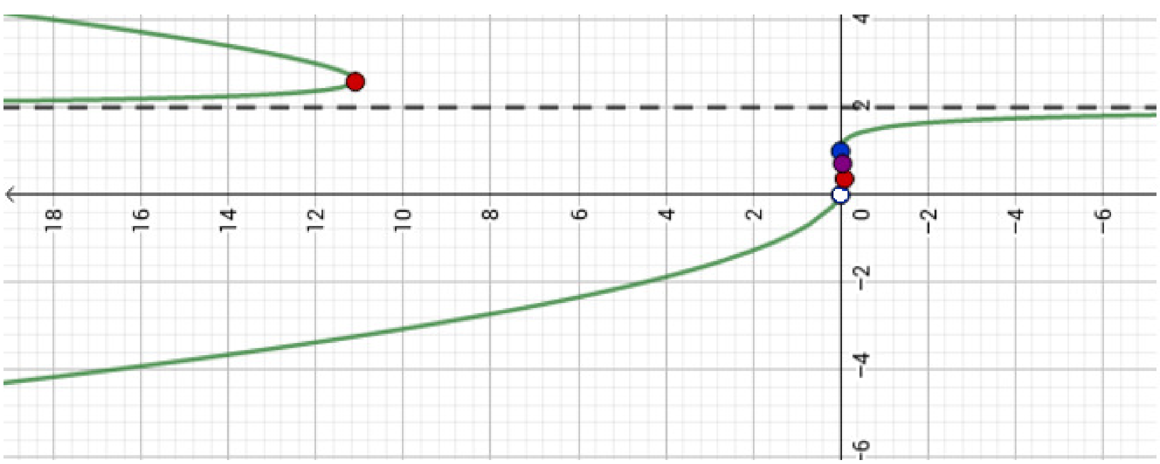
9.
$$f''(x) = \left(x^2 + 1 + \frac{2}{x-2}\right)'' = \left(2x - \frac{2}{(x-2)^2}\right)' = 2 + \frac{4}{(x-2)^3} = 2\frac{(x-2)^3 + 2}{(x-2)^3}.$$

The equation $(x-2)^3+2 = 0$ is equivalent to $x-2 = -\sqrt[3]{2}$, and has only one real solution: $x = 2 - \sqrt[3]{2} \approx 0.74$. So, f'' has only one root, $x = 2 - \sqrt[3]{2} \approx 0.74$. Since this root is simple, f'' changes sign at it. Also, f'' changes sign at the discontinuity point $x = 2$. At the removable discontinuity point $x = 0$ nothing happens. For large x, $f''(x)$ is clearly positive. This information is sufficient for constructing the sign table of f'':

interval	$(-\infty, 2-\sqrt[3]{2})$	$(2-\sqrt[3]{2}, 2)$	$(2, +\infty)$
sign of f''	$+$	$-$	$+$
concavity of f	up	down	up

From the table, we see that f has only one inflection point, $x = 2 - \sqrt[3]{2}$.

10. Summarising all the available information, we can sketch the graph of f:



On this graph you see

- the vertical asymptote $x = 2$;
- parabolic shape for large and small x;
- the cut at $(0, 0)$;
- the root (and local maximum) at $x = 1$ (in blue);
- the local minima at $x = \frac{3\pm\sqrt{5}}{2}$ (in red);
- the inflection point at $x = 2 - \sqrt[3]{2}$ (in violet).