Homework/Tutorial 7

A complete solution to questions 1 to 4 is worth 1 mark; for questions 5 to 7 it is 2 marks.

What this homework is about

You will compute derivatives and use them to analyse functions.

Reminder

Differentiation rules

$$c' = 0, \qquad (x^r)' = rx^{r-1},\ r \in \mathbb{R}$$

$$\sin' = \cos, \quad \cos' = -\sin, \quad \tan' = \frac{1}{\cos^2}, \quad \cot' = -\frac{1}{\sin^2}$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} = -\arccos'(x)$$

$$\arctan'(x) = \frac{1}{1+x^2} = -\operatorname{arccot}'(x)$$

$$(f \pm g)' = f' \pm g', \quad (cf)' = cf'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}, \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f \circ g)'(x) = f'(g(x))g'(x), \qquad (f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

Applications of differential calculus

Different properties of a function can be established by looking at its derivative. Here are some examples:

1. $f'(x) = 0$ on (a, b) $\iff$ $f(x) = c$ on (a, b) for some $c \in \mathbb{R}$.
2. $f'(x) = g'(x)$ on (a, b) $\iff$ $f(x) = g(x) + c$ on (a, b) for some $c \in \mathbb{R}$.
3. $f^{(n)}(x) = 0$ on (a, b) $\iff$ $f(x)$ is polynomial of degree $< n$ on (a, b).
4. f is differentiable and increasing (resp. decreasing) on (a, b) $\implies$ $f'(x) \geq 0$ (resp. ≤ 0) on (a, b).
5. $f'(x) > 0$ (resp. < 0) on (a, b) $\implies$ f is increasing (resp. decreasing) on (a, b).
6. f is defined on (a, b) and has a local extremum at c $\implies$ c is critical.

A point c from the domain of a function f is called

- **critical** if f is not differentiable at c or if $f'(c) = 0$;
- **stationary** if $f'(c) = 0$;
- a point of **local minimum (resp. maximum)** if $f(x) \geq f(c)$ (resp. $\leq f(c)$) for all x sufficiently close to x;
- a point of **global minimum (resp. maximum)** if $f(x) \geq f(c)$ (resp. $\leq f(c)$) for all x in the domain of f.

Extremum means minimum or maximum.

Questions

1. Compute the derivative of the following function: $\dfrac{x^2-1}{x^3-x^2+2x-2}$.

   Solution. Let us first simplify the function:
$$\frac{x^2-1}{x^3-x^2+2x-2} = \frac{(x-1)(x+1)}{(x-1)(x^2+2)} = \frac{x+1}{x^2+2}$$
   for $x \neq 1$. Now we can use the quotient rule for derivatives:
$$\begin{aligned}\left(\frac{x^2-1}{x^3-x^2+2x-2}\right)' = \left(\frac{x+1}{x^2+2}\right)' &= \frac{(x+1)'(x^2+2)-(x+1)(x^2+2)'}{(x^2+2)^2} \\ &= \frac{1\cdot(x^2+2)-(x+1)\cdot 2x}{(x^2+2)^2} = \frac{x^2+2-2x^2-2x}{(x^2+2)^2} \\ &= \frac{-x^2-2x+2}{(x^2+2)^2}, \qquad x \neq 1.\end{aligned}$$

2. Compute the second derivative of the following function: $\tan(\frac{x}{2})$.

   Solution.
$$\begin{aligned}\tan(\frac{x}{2})'' = (\tan(\frac{x}{2})')' &= \left(\frac{1}{(\cos(\frac{x}{2}))^2}\cdot\left(\frac{x}{2}\right)'\right)' = \frac{1}{2}\left((\cos(\frac{x}{2}))^{-2}\right)' \\ &= \frac{1}{2}\cdot(-2)(\cos(\frac{x}{2}))^{-3}\cdot(\cos(\frac{x}{2}))' = -(\cos(\frac{x}{2}))^{-3}\cdot(-\sin(\frac{x}{2}))\cdot\left(\frac{x}{2}\right)' \\ &= \frac{\sin(\frac{x}{2})}{2\cos(\frac{x}{2})^3}.\end{aligned}$$
   We see that our function is differentiable 2 times on its domain, i.e., for $x \neq (2k+1)\pi$ for integers k. Indeed, the only possible problem here is division by zero, which occurs when $\cos(\frac{x}{2}) = 0$, i.e., $\frac{x}{2} = \frac{\pi}{2} + k\pi$.

3. Assume that a function f satisfies $f''(x) = \cos(x)$. Show that f is of the form $f(x) = ax + b - \cos(x)$ for some real a and b.

   Solution. We have $f''(x) = \cos(x) = (-\cos(x))''$, so $(f(x)+\cos(x))'' = 0$. By a theorem from lectures, this implies that $f(x)+\cos(x)$ is a polynomial of degree < 2, and thus has the form $ax+b$. So, $f(x)+\cos(x) = ax+b$, as desired.

4. Consider the function $f(x) = \begin{cases} x, & x \leq 0; \\ \sin(x), & x > 0. \end{cases}$
   (a) Compute its first and second derivatives.
   (b) Determine all critical and stationary points of f''.

   Solution.
   - For $x < 0$, we have $f'(x) = x' = 1$, $f''(x) = 1' = 0$, $f'''(x) = 0' = 0$.
   - For $x > 0$, we have $f'(x) = \sin(x)' = \cos(x)$, $f''(x) = \cos(x)' = -\sin(x)$, $f'''(x) = (-\sin(x))' = -\cos(x)$.
   - The same formulas give one-sided derivatives at 0:
     $f'_-(0) = 1$, $f'_+(0) = \cos(0) = 1$, so $f'(0) = 1$ (we used that $f(x) = \sin(x)$ even for $x \geq 0$, since $\sin(0) = 0$);
     $f''_-(0) = 0$, $f''_+(0) = -\sin(0) = 0$, so $f''(0) = 0$;

$f'''_-(0) = 0$, $f'''_+(0) = -\cos(0) = -1$, so f is differentiable only twice at 0.

From the above, we conclude:

(a) $f'(x) = \begin{cases} 1, & x \le 0; \\ \cos(x), & x > 0; \end{cases}$ $\quad f''(x) = \begin{cases} 0, & x \le 0; \\ -\sin(x), & x > 0. \end{cases}$

(b) The stationary points of f'' are those x for which $(f'')'(x) = f'''(x)$ is zero. These are all $x < 0$, and all x of the form $\frac{\pi}{2} + k\pi$ for non-negative integers k (this is where $-\cos(x)$ vanishes). Critical points include all the stationary points and $x = 0$, which is the only point at which $f'''(x)$ does not exist.

5. Consider the function $g(x) = \arccos(\sin(x)^2)$.
   (a) Compute g'. At what x is g not differentiable?
   (b) For g', determine its one-sided limits and discontinuity type at $x = \frac{\pi}{2}$.

Solution.

(a)

$$g'(x) = -\frac{1}{\sqrt{1-(\sin(x)^2)^2}} \cdot (\sin(x)^2)' = -\frac{1}{\sqrt{1-\sin(x)^4}} \cdot 2\sin(x) \cdot (\sin(x))'$$
$$= \frac{-2\sin(x)\cos(x)}{\sqrt{1-\sin(x)^4}}.$$

One can leave the answer as it is, or slightly simplify it:

$$\frac{-2\sin(x)\cos(x)}{\sqrt{1-\sin(x)^4}} = \frac{-2\sin(x)\cos(x)}{\sqrt{(1-\sin(x)^2)(1+\sin(x)^2)}} = \frac{-2\sin(x)\cos(x)}{\sqrt{\cos(x)^2}\sqrt{1+\sin(x)^2}}$$
$$= \frac{\cos(x)}{|\cos(x)|}\frac{-2\sin(x)}{\sqrt{1+\sin(x)^2}} = \begin{cases} \dfrac{-2\sin(x)}{\sqrt{1+\sin(x)^2}} & \text{when } \cos(x) > 0; \\ \dfrac{2\sin(x)}{\sqrt{1+\sin(x)^2}} & \text{when } \cos(x) < 0. \end{cases}$$

When splitting the square root into two, we used that $1 \pm \sin(x)^2 \ge 0$ for all x.

Our function g is not differentiable when $\sqrt{1-\sin(x)^4} = 0$, i.e., $\sin(x) = \pm 1$. This happens precisely when $x = \frac{\pi}{2} + k\pi$ for some integer k.

(b) Since $\cos(x) > 0$ for x slightly smaller than $\frac{\pi}{2}$ and $\cos(x) < 0$ for x slightly bigger than $\frac{\pi}{2}$, the above formulas yield

$$g'_-(\frac{\pi}{2}) = \lim_{x\to(\frac{\pi}{2})^-} \frac{-2\sin(x)}{\sqrt{1+\sin(x)^2}} = \frac{-2\cdot 1}{\sqrt{1+1^2}} = -\sqrt{2};$$
$$g'_+(\frac{\pi}{2}) = \lim_{x\to(\frac{\pi}{2})^+} \frac{2\sin(x)}{\sqrt{1+\sin(x)^2}} = \frac{2\cdot 1}{\sqrt{1+1^2}} = \sqrt{2}.$$

We used $\sin(\frac{\pi}{2}) = 1$.

Since $g'_-(\frac{\pi}{2})$ and $g'_+(\frac{\pi}{2})$ exist and are distinct, g' has a jump discontinuity at $x = \frac{\pi}{2}$.

6. Consider the function $h(x) = x^3 + 2x + 1$.
   (a) Compute h'. Use this to show that h has an inverse function.
   (b) Compute $(h^{-1})'(1)$.

Solution.

(a) $h'(x) = 3x^2 + 2$. Since $h'(x) > 0$ for all x, the function h is increasing on $\mathbb{R}$, hence one-to-one, hence invertible (using the results we saw in the chapter Inverse Functions).

(b) $(h^{-1})'(1) = \dfrac{1}{h'(h^{-1}(1))}$. We need to compute $h^{-1}(1)$, that is, the value of x for which $x^3 + 2x + 1 = 1$. Clearly, this value is $x = 0$. So,

$$(h^{-1})'(1) = \frac{1}{h'(0)} = \frac{1}{3 \cdot 0^2 + 2} = \frac{1}{2}.$$

7. Consider the curve described by the equation $x^2 + y^2 = xy - x - y + 6$.
   (a) Check that the point $(1, 2)$ lies on this curve.
   (b) What is the equation of its tangent line at the point $(1, 2)$?

Solution.

(a) $1^2 + 2^2 = 5 = 1 \cdot 2 - 1 - 2 + 6$.

(b) Let us differentiate both sides of our equation:

$$\begin{aligned} & (x^2 + y^2)' = (xy - x - y + 6)' \\ \iff \quad & 2x + 2yy' = y + xy' - 1 - y' \\ \iff \quad & y'(2y - x + 1) = y - 2x - 1 \\ \implies \quad & y' = \frac{y - 2x - 1}{2y - x + 1}. \end{aligned}$$

Putting $x = 1$ and $y = 2$ in the formula, we get the slope of the tangent line:

$$y' = \frac{2 - 2 - 1}{4 - 1 + 1} = -\frac{1}{4}.$$

The point-slope formula then yields the equation of the desired tangent line:

$$\begin{aligned} & y = -\frac{1}{4}(x - 1) + 2, \\ \text{or} \qquad & y = -\frac{x}{4} + \frac{9}{4}, \\ \text{or} \qquad & y = -0.25x + 2.25. \end{aligned}$$