Homework/Tutorial 7

A complete solution to questions 1 to 4 is worth 1 mark; for questions 5 to 7 it is 2 marks.

What this homework is about
You will compute derivatives and use them to analyse functions.

Reminder

Differentiation rules

\[ \begin{align*}
  c' &= 0, & (x^r)' &= r x^{r-1}, & r \in \mathbb{R} \\
  \sin' &= \cos, & \cos' &= -\sin, & \tan' &= \frac{1}{\cos^2}, & \cot' &= -\frac{1}{\sin^2} \\
  (\arcsin(x))' &= \frac{1}{\sqrt{1-x^2}}, \quad & (\arccos(x))' &= -\frac{1}{\sqrt{1-x^2}} \\
  (\arctan(x))' &= \frac{1}{1+x^2}, \quad & (\arccot(x))' &= -\frac{1}{1+x^2} \\
  (f \pm g)' &= f' \pm g', & (cf)' &= cf' \\
  (fg)' &= f'g + fg' \\
  (f^{-1})' &= \frac{1}{f' \circ f^{-1}} \\
\end{align*} \]

Applications of differential calculus
Different properties of a function can be established by looking at its derivative. Here are some examples:

1. \( f'(x) = 0 \) on \((a, b)\) \iff \( f(x) = c \) on \((a, b)\) for some \( c \in \mathbb{R} \).
2. \( f'(x) = g'(x) \) on \((a, b)\) \iff \( f(x) = g(x) + c \) on \((a, b)\) for some \( c \in \mathbb{R} \).
3. \( f^{(n)}(x) = 0 \) on \((a, b)\) \iff \( f(x) \) is polynomial of degree \(< n \) on \((a, b)\).
4. \( f \) is differentiable and increasing (resp. decreasing) on \((a, b)\) \implies \( f'(x) \geq 0 \) (resp. \( \leq 0 \)) on \((a, b)\).
5. \( f'(x) > 0 \) (resp. \( < 0 \)) on \((a, b)\) \implies \( f \) is increasing (resp. decreasing) on \((a, b)\).
6. \( f \) is defined on \((a, b)\) and has a local extremum at \( c \) \implies \( c \) is critical.

A point \( c \) from the domain of a function \( f \) is called

- **critical** if \( f \) is not differentiable at \( c \) or if \( f'(c) = 0 \);
- **stationary** if \( f'(c) = 0 \);
- a point of **local minimum** (resp. **maximum**) if \( f(x) \geq f(c) \) (resp. \( \leq f(c) \)) for all \( x \) sufficiently close to \( c \);
- a point of **global minimum** (resp. **maximum**) if \( f(x) \geq f(c) \) (resp. \( \leq f(c) \)) for all \( x \) in the domain of \( f \).

**Extremum** means minimum or maximum.
Questions

1. Compute the derivative of the following function: \( \frac{x^2-1}{x^3-x^2+2x-2} \).

\textbf{Solution.} Let us first simplify the function:
\[
\frac{x^2-1}{x^3-x^2+2x-2} = \frac{(x-1)(x+1)}{(x-1)(x^2+2)} = \frac{x+1}{x^2+2}
\]
for \( x \neq 1 \). Now we can use the quotient rule for derivatives:
\[
\left( \frac{x^2-1}{x^3-x^2+2x-2} \right)' = \left( \frac{x+1}{x^2+2} \right)' = \frac{(x+1)'(x^2+2) - (x+1)(x^2+2)'}{(x^2+2)^2}
\]
\[
= \frac{1 \cdot (x^2+2) - (x+1) \cdot 2x}{(x^2+2)^2} = \frac{x^2+2-2x^2-2x}{(x^2+2)^2} = \frac{-x^2-2x+2}{(x^2+2)^2}, \quad x \neq 1.
\]

2. Compute the second derivative of the following function: \( \tan\left(\frac{x}{2}\right) \).

\textbf{Solution.}
\[
\tan\left(\frac{x}{2}\right)'' = (\tan\left(\frac{x}{2}\right))' = \left( \frac{1}{(\cos\left(\frac{x}{2}\right))^2} \cdot \left( \frac{x}{2} \right)' \right)' = \frac{1}{2} \left( (\cos\left(\frac{x}{2}\right))^{-3} \cdot (\cos\left(\frac{x}{2}\right))' - (\cos\left(\frac{x}{2}\right))^{-3} \cdot (-\sin\left(\frac{x}{2}\right)) \cdot \left( \frac{x}{2} \right)' \right)
\]
\[
= \frac{\sin\left(\frac{x}{2}\right)}{2\cos^3\left(\frac{x}{2}\right)}.
\]
We see that our function is differentiable 2 times on its domain, i.e., for \( x \neq (2k+1)\pi \) for integers \( k \). Indeed, the only possible problem here is division by zero, which occurs when \( \cos\left(\frac{x}{2}\right) = 0 \), i.e., \( \frac{x}{2} = \frac{x}{2} + k\pi \).

3. Assume that a function \( f \) satisfies \( f''(x) = \cos(x) \). Show that \( f \) is of the form \( f(x) = ax + b - \cos(x) \) for some real \( a \) and \( b \).

\textbf{Solution.} We have \( f''(x) = \cos(x) = (-\cos(x))'' \), so \( (f(x) + \cos(x))'' = 0 \). By a theorem from lectures, this implies that \( f(x) + \cos(x) \) is a polynomial of degree \(< 2\), and thus has the form \( ax + b \). So, \( f(x) + \cos(x) = ax + b \), as desired.

4. Consider the function \( f(x) = \begin{cases} x, & x \leq 0; \\ \sin(x), & x > 0. \end{cases} \)

\textbf{(a)} Compute its first and second derivatives.
\textbf{(b)} Determine all critical and stationary points of \( f'' \).

\textbf{Solution.}

- For \( x < 0 \), we have \( f'(x) = x' = 1, f''(x) = 1' = 0, f'''(x) = 0' = 0 \).
- For \( x > 0 \), we have \( f'(x) = \sin(x)' = \cos(x), f''(x) = \cos(x)' = -\sin(x), f'''(x) = (-\sin(x))' = -\cos(x) \).
- The same formulas give one-sided derivatives at 0:
  \( f'_- (0) = 1, f'_+ (0) = \cos(0) = 1, \) so \( f'(0) = 1 \) (we used that \( f(x) = \sin(x) \) even for \( x \geq 0 \), since \( \sin(0) = 0 \));
  \( f''_- (0) = 0, f''_+ (0) = -\sin(0) = 0, \) so \( f''(0) = 0 \);
5. Consider the function
\[ f''(0) = 0, \quad f'''(0) = -\cos(0) = -1, \] so \( f \) is differentiable only twice at 0.

From the above, we conclude:

(a) \( f'(x) = \begin{cases} 1, & x \leq 0; \\
\cos(x), & x > 0; \end{cases} \quad f''(x) = \begin{cases} 0, & x \leq 0; \\
-\sin(x), & x > 0. \end{cases} \)

(b) The stationary points of \( f'' \) are those \( x \) for which \( (f'')'(x) = f'''(x) \) is zero. These are all \( x < 0 \), and all \( x \) of the form \( \frac{\pi}{2} + k\pi \) for non-negative integers \( k \) (this is where \( -\cos(x) \) vanishes). Critical points include all the stationary points and \( x = 0 \), which is the only point at which \( f'''(x) \) does not exist.

5. Consider the function \( g(x) = \arccos(\sin(x)^2) \).

(a) Compute \( g' \). At what \( x \) is \( g \) not differentiable?

(b) For \( g' \), determine its one-sided limits and discontinuity type at \( x = \frac{\pi}{2} \).

Solution.

(a) \[
g'(x) = -\frac{1}{\sqrt{1 - (\sin(x))^2}} \cdot (\sin(x))^2' = -\frac{1}{\sqrt{1 - \sin(x)^4}} \cdot 2\sin(x) \cdot (\sin(x))'
\]

One can leave the answer as it is, or slightly simplify it:

\[
\frac{-2\sin(x) \cos(x)}{\sqrt{1 - \sin(x)^4}} = \frac{-2\sin(x) \cos(x)}{\sqrt{(1 - \sin(x)^2)(1 + \sin(x)^2)}} = \frac{-2\sin(x) \cos(x)}{\sqrt{\cos(x)^2 \sqrt{1 + \sin(x)^2}}}
\]

\[
= \frac{\cos(x)}{\sqrt{1 + \sin(x)^2}} \frac{-2\sin(x)}{\sqrt{1 + \sin(x)^2}} = \begin{cases} 
-2\sin(x) \\
\sqrt{1 + \sin(x)^2} \\
\sqrt{1 + \sin(x)^2}
\end{cases}
\]

when \( \cos(x) > 0 \); \( \frac{\cos(x)}{\sqrt{1 + \sin(x)^2}} \)

When splitting the square root into two, we used that \( 1 \pm \sin(x)^2 \geq 0 \) for all \( x \).

Our function \( g \) is not differentiable when \( \sqrt{1 - \sin(x)^4} = 0 \), i.e., \( \sin(x) = \pm 1 \). This happens precisely when \( x = \frac{\pi}{2} + k\pi \) for some integer \( k \).

(b) Since \( \cos(x) > 0 \) for \( x \) slightly smaller than \( \frac{\pi}{2} \) and \( \cos(x) < 0 \) for \( x \) slightly bigger than \( \frac{\pi}{2} \), the above formulas yield

\[
g'_{-}(\frac{\pi}{2}) = \lim_{x \to (\frac{\pi}{2})^-} \frac{-2\sin(x)}{\sqrt{1 + \sin(x)^2}} = \frac{-2 \cdot 1}{\sqrt{1 + 1}} = -\sqrt{2};
\]

\[
g'_{+}(\frac{\pi}{2}) = \lim_{x \to (\frac{\pi}{2})^+} \frac{2\sin(x)}{\sqrt{1 + \sin(x)^2}} = \frac{2 \cdot 1}{\sqrt{1 + 1}} = \sqrt{2}.
\]

We used \( \sin(\frac{\pi}{2}) = 1 \).

Since \( g'_{-}(\frac{\pi}{2}) \) and \( g'_{+}(\frac{\pi}{2}) \) exist and are distinct, \( g' \) has a jump discontinuity at \( x = \frac{\pi}{2} \).

6. Consider the function \( h(x) = x^3 + 2x + 1 \).

(a) Compute \( h' \). Use this to show that \( h \) has an inverse function.

(b) Compute \( (h^{-1})'(1) \).

Solution.

(a) \( h'(x) = 3x^2 + 2 \). Since \( h'(x) > 0 \) for all \( x \), the function \( h \) is increasing on \( \mathbb{R} \),
hence one-to-one, hence invertible (using the results we saw in the chapter Inverse Functions).

(b) \((h^{-1})'(1) = \frac{1}{h'(h^{-1}(1))}\). We need to compute \(h^{-1}(1)\), that is, the value of \(x\) for which \(x^3 + 2x + 1 = 1\). Clearly, this value is \(x = 0\). So,
\[
(h^{-1})'(1) = \frac{1}{h'(0)} = \frac{1}{3 \cdot 0^2 + 2} = \frac{1}{2}.
\]

7. Consider the curve described by the equation \(x^2 + y^2 = xy - x - y + 6\).

(a) Check that the point \((1, 2)\) lies on this curve.

(b) What is the equation of its tangent line at the point \((1, 2)\)?

**Solution.**

(a) \(1^2 + 2^2 = 5 = 1 \cdot 2 - 1 - 2 + 6\).

(b) Let us differentiate both sides of our equation:
\[
(x^2 + y^2)' = (xy - x - y + 6)'
\]
\[
\Longleftrightarrow 2x + 2yy' = y + xy' - 1 - y'
\]
\[
\Longleftrightarrow y'(2y - x + 1) = y - 2x - 1
\]
\[
\Longrightarrow y' = \frac{y - 2x - 1}{2y - x + 1}.
\]

Putting \(x = 1\) and \(y = 2\) in the formula, we get the slope of the tangent line:
\[
y' = \frac{2 - 2 - 1}{4 - 1 + 1} = -\frac{1}{4}.
\]

The point-slope formula then yields the equation of the desired tangent line:
\[
y = -\frac{1}{4}(x - 1) + 2,
\]
\[
or \quad y = -\frac{x}{4} + \frac{9}{4},
\]
\[
\text{or} \quad y = -0.25x + 2.25.
\]