Homework/Tutorial 7

A complete solution to questions 1 to 4 is worth 1 mark; for questions 5 to 7 it is 2 marks.

What this homework is about

You will compute derivatives and use them to analyse functions.

Reminder

Differentiation rules

$$c' = 0, \qquad (x^r)' = rx^{r-1},\ r \in \mathbb{R}$$

$$\sin' = \cos, \quad \cos' = -\sin, \quad \tan' = \frac{1}{\cos^2}, \quad \cot' = -\frac{1}{\sin^2}$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} = -\arccos'(x)$$

$$\arctan'(x) = \frac{1}{1+x^2} = -\operatorname{arccot}'(x)$$

$$(f \pm g)' = f' \pm g', \quad (cf)' = cf'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}, \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f \circ g)'(x) = f'(g(x))g'(x), \qquad (f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

Applications of differential calculus

Different properties of a function can be established by looking at its derivative. Here are some examples:

1. $f'(x) = 0$ on (a,b) $\iff$ $f(x) = c$ on (a,b) for some $c \in \mathbb{R}$.
2. $f'(x) = g'(x)$ on (a,b) $\iff$ $f(x) = g(x) + c$ on (a,b) for some $c \in \mathbb{R}$.
3. $f^{(n)}(x) = 0$ on (a,b) $\iff$ $f(x)$ is polynomial of degree $< n$ on (a,b).
4. f is differentiable and increasing (resp. decreasing) on (a,b) $\implies$ $f'(x) \geq 0$ (resp. ≤ 0) on (a,b).
5. $f'(x) > 0$ (resp. < 0) on (a,b) $\implies$ f is increasing (resp. decreasing) on (a,b).
6. f is defined on (a,b) and has a local extremum at c $\implies$ c is critical.

A point c from the domain of a function f is called

- **critical** if f is not differentiable at c or if $f'(c) = 0$;
- **stationary** if $f'(c) = 0$;
- a point of **local minimum (resp. maximum)** if $f(x) \geq f(c)$ (resp. $\leq f(c)$) for all x sufficiently close to x;
- a point of **global minimum (resp. maximum)** if $f(x) \geq f(c)$ (resp. $\leq f(c)$) for all x in the domain of f.

Extremum means minimum or maximum.

Questions

1. Compute the derivative of the following function: $\dfrac{x^2-1}{x^3-x^2+2x-2}$.
2. Compute the second derivative of the following function: $\tan(\frac{x}{2})$.
3. Assume that a function f satisfies $f''(x) = \cos(x)$. Show that f is of the form $f(x) = ax + b - \cos(x)$ for some real a and b.
4. Consider the function $f(x) = \begin{cases} x, & x \leq 0; \\ \sin(x), & x > 0. \end{cases}$.
   (a) Compute its first and second derivatives.
   (b) Determine all critical and stationary points of f''.
5. Consider the function $g(x) = \arccos(\sin(x)^2)$.
   (a) Compute g'. At what x is g not differentiable?
   (b) For g', determine its one-sided limits and discontinuity type at $x = \frac{\pi}{2}$.
6. Consider the function $h(x) = x^3 + 2x + 1$.
   (a) Compute h'. Use this to show that h has an inverse function.
   (b) Compute $(h^{-1})'(1)$.
7. Consider the curve described by the equation $x^2 + y^2 = xy - x - y + 6$.
   (a) Check that the point $(1, 2)$ lies on this curve.
   (b) What is the equation of its tangent line at the point $(1, 2)$?