Homework/Tutorial 6

A complete solution to questions 1 and 2 is worth 3 marks; for question 3 it is 1.4 marks; for the remaining questions it is 1.3 marks.

What this homework is about

You will learn how to apply the Intermediate Value Theorem, and how to check the continuity and compute the derivatives of the simplest functions.

Reminder

Intermediate Value Theorem (IVT). If a function f is continuous on the closed interval $[a, b]$, then it takes every real value between $f(a)$ and $f(b)$.

If a composition $f \circ g$ of two continuous functions f and g is defined on an interval (a, b), that it is itself continuous on (a, b).

If f is one-to-one and continuous on an interval or a ray, then its inverse f^{-1} is continuous on its domain.

Given a function f, the function f' defined by the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is called the **derivative of f with respect to** x. The domain of f' consists of all x for which the limit exists. The function f is said to be **differentiable at** x_0 if the limit above exists for $x = x_0$. When only the one-sided version $\lim\limits_{h\to 0^\pm}$ of the above limit is defined, we talk about **left-hand** and **right-hand derivatives** $f'_\pm(x)$.

The equation of the **tangent line** to the graph of a function f at the point x_0 (where f is differentiable) can be written as

$$y = f'(x_0)(x - x_0) + f(x_0).$$

A function $f(x)$ differentiable at $x = x_0$ is continuous at $x = x_0$, but the converse does not always hold.

A summary of differentiation rules:

$$(c)' = 0, \qquad (f \pm g)' = f' \pm g',$$
$$(x^r)' = rx^{r-1}, \qquad (fg)' = f'g + fg'.$$

Here c and r are any real numbers, and f and g are functions differentiable at the points of interest.

Questions

1. Consider the function $f(x) = \arcsin(|2x + 1| - 2)$.
   (a) What is its natural domain?
   (b) Explain why f is continuous on its natural domain.
   (c) Show that the graph of f intersects the line $y = -\frac{x}{2}$ at least twice. (*Hint.* Use the Intermediate Value Theorem.)

   Solution.

(a) The function $h(x) = |2x+1| - 2$ is defined everywhere, and the function arcsin is defined on $[-1,1]$. So, we need to determine all x for which $-1 \leq |2x+1| - 2 \leq 1$, that is $1 \leq |2x+1| \leq 3$.
- When $2x+1 \geq 0$, this means $1 \leq 2x+1 \leq 3$, that is $0 \leq x \leq 1$.
- When $2x+1 < 0$, this means $1 \leq -2x-1 \leq 3$, that is $-2 \leq x \leq -1$.

Summarising, the natural domain is $[-2,-1] \sqcup [0,1]$.

(b)
- The function $g_2(x) = |x|$ is continuous everywhere (it is continuous on $(0,+\infty)$ and $(-\infty,0)$ since it coincides with the functions x ans $-x$ on these rays; and it is continuous at 0 since $\lim\limits_{x\to 0^+} x = \lim\limits_{x \to 0^-}(-x) = 0$.
- The functions $g_1(x) = 2x+1$ and $g_3(x) = x-2$ are polynomial and hence continuous everywhere.
- The function arcsin is continuous on $[-1,1]$ as the inverse of the continuous function sin (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

So, the iterated composition $f = \arcsin \circ g_3 \circ g_2 \circ g_1$ is continuous on its domain.

(c) The x-coordinates of the intersection points of the curve $y = \arcsin(|2x+1|-2)$ and the line $y = -\frac{x}{2}$ are the solutions of equation $\arcsin(|2x+1|-2) = -\frac{x}{2}$, that is, $u(x) = 0$, where $u(x) = \arcsin(|2x+1|-2) + \frac{x}{2}$. We saw above that the function f, and hence u, is continuous on $[-2,-1]$. Also, we have $u(-2) = \arcsin(1) - 1 = \frac{\pi}{2} - 1 > 0$, and $u(-1) = \arcsin(-1) - \frac{1}{2} = -\frac{\pi}{2} - \frac{1}{2} < 0$. According to the Intermediate Value Theorem, this means that u takes the value 0 somewhere between -2 and -1. One similarly shows that it takes the value 0 between 0 and 1. This gives two distinct solutions of $u(x) = 0$, hence two intersection points.

2. Consider the functions

$$f(x) = \cos(\frac{1}{x}), \qquad g(x) = x\sqrt[3]{x}\cos(\frac{1}{x}).$$

(a) What are their natural domains?

(b) Compute the value of f at $x = \frac{1}{k\pi}$ for non-zero integer values of k. (The answer might depend on k.)

(c) Does f have a limit at 0?

(d) What is the discontinuity type of f at the point 0?

(e) Is the function g even? odd?

(f) What is the discontinuity type of g at the point 0?

(g) Explain (briefly) why f and g are continuous on their natural domains.

(h) Consider the function $h(x) = \begin{cases} g(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Show that it is differentiable at 0.

Solution.

(a) The only problematic expression for both functions is $\frac{1}{x}$, defined for $x \neq 0$ only. So, the natural domain for both functions is $(-\infty,0) \cup (0,+\infty)$.

(b) $f(\frac{1}{k\pi}) = \cos(k\pi) = \begin{cases} 1, & x \text{ even and } x \neq 0, \\ -1, & x \text{ odd}. \end{cases}$.

(c) The points $x = \frac{1}{k\pi}$ get as close as we wish to 0 both for even and for odd k. According to the previous question, $\lim\limits_{x\to 0} f(x)$ then has to be 1 and -1 simultaneously, which is impossible. So f does not have a limit at 0.

(d) Oscillating discontinuity.

(e) The natural domain of g is $(-\infty,0) \cup (0,+\infty)$, which is symmetric with respect to

0. Further, $g(-x) = (-x)\sqrt[3]{-x}\cos(\frac{1}{-x}) = -x(-\sqrt[3]{x})\cos(\frac{1}{x}) = x\sqrt[3]{x}\cos(\frac{1}{x}) = g(x)$ (we used that cos is an even function). Conclusion: the function g is even.

(f) Removable discontinuity. Indeed, the factor $\cos(\frac{1}{x})$ is bounded (since $|\cos(t)| \leq 1$ for all t), and $x\sqrt[3]{x} \underset{x\to 0}{\to} 0$. If something which gets as small as you wish when x approaches 0 is multiplied by something bounded, the result is still as small as you wish. So, g has a finite limit at 0: $\lim\limits_{x\to 0} g(x) = 0$.

Remark. If you have read the supplementary materials, you can compute $\lim\limits_{x\to 0} g(x)$ more rigorously using the following result:

Squeeze Theorem. If the functions f, g, and h are such that

i. $g(x) \leq f(x) \leq h(x)$ for all x in an interval containing c (possibly excluding c),

ii. $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$,

then $\lim\limits_{x\to c} f(x) = L$ as well.

(g) They are both compositions and products of continuous functions.

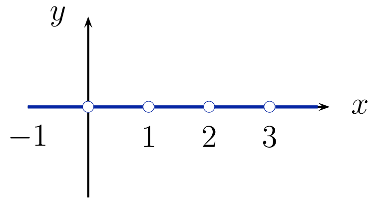
(h) $\lim\limits_{x\to 0} \dfrac{h(x) - h(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{g(x)}{x} = \lim\limits_{x\to 0} \sqrt[3]{x}\cos(\frac{1}{x}) = 0$ We used the identity $h(x) = g(x)$ for $x \neq 0$, and, in the last step, the same boundedness argument as in (f).

3. Recall that the *floor* $\lfloor x \rfloor$ of a real number x is defined as the greatest integer that is less than or equal to x. Compute the left-hand and the right-hand derivatives of the function $f(x) = \lfloor x \rfloor$. Plot the graph of f' and determine the type of its discontinuity points.

Solution. For all integers k, one has $\lfloor x \rfloor = k$ for $x \in [k, k+1)$, so our function is constant on each half-closed interval $[k, k+1)$. So,

- $f'(x) = 0$ for all x inside such intervals, that is, for all non-integer x;
- $f'_+(k) = 0$ for all integers k.

Finally, $f'_-(k) = \lim\limits_{x\to k^-} \dfrac{\lfloor x \rfloor - \lfloor k \rfloor}{x - k} = \lim\limits_{x\to k^-} \dfrac{(k-1) - k}{x - k} = \lim\limits_{x\to k^-} \dfrac{-1}{x - k} = +\infty$. We used that $\lfloor x \rfloor = k - 1$ for x sufficiently close to k and smaller than k. So, f has neither left-hand nor two-sided derivative at $x = k$. The graph of f' looks as follows:



We see that f' has a removable continuity at each integer point $x = k$.

4. Find the equation of the tangent line to the graph of the function

$$f(x) = (x^2 + 1 + \frac{1}{(x-2)^3})(x^3 - \sqrt{x})$$

at the point $x = 1$.

Solution. We need to compute two things:

(a) $f(1) = (1 + 1 - 1)(1 - 1) = 0$;

(b) $f'(1) = \Big((x^2 + 1 + \frac{1}{(x-2)^3})(x^3 - \sqrt{x})\Big)' |_{x=1}$

$= \Big((x^2 + 1 + \frac{1}{(x-2)^3})'(x^3 - \sqrt{x}) + (x^2 + 1 + \frac{1}{(x-2)^3})(x^3 - \sqrt{x})'\Big) |_{x=1}$

$= \Big((x^2 + 1 + \frac{1}{(x-2)^3})'(x^3 - \sqrt{x}) + (x^2 + 1 + \frac{1}{(x-2)^3})(3x^2 - \frac{1}{2\sqrt{x}})\Big) |_{x=1}$

$= (x^2 + 1 + \frac{1}{(x-2)^3})'|_{x=1}(1-1) + (1+1-1)(3-\frac{1}{2})$
$= 0 + \frac{5}{2} = \frac{5}{2}.$

Now, the equation of the tangent line is $y = f'(1)(x-1) + f(1) = \frac{5}{2}(x-1) + 0$, that is, $y = \frac{5}{2}(x-1)$.

5. Compute the area of the triangle formed by the x-axis, the y-axis, and the tangent line to the hyperbola $y = \frac{1}{x}$ at the point $x = a$, where a is a positive real number. The answer might depend on a.

   Solution. We need the equation of the tangent line to the graph of $f(x) = \frac{1}{x}$ at $x = a$:
   $$y = f'(a)(x-a) + f(a) = -\frac{1}{a^2}(x-a) + \frac{1}{a} = -\frac{x}{a^2} + \frac{1}{a} + \frac{1}{a},$$
   that is, $y = -\frac{x}{a^2} + \frac{2}{a}$. It intersects
   - the y-axis at the point $(0, \frac{2}{a})$ (since $f(0) = \frac{2}{a}$);
   - the x-axis at the point $(2a, 0)$ (to determine this, we solved the equation $-\frac{x}{a^2} + \frac{2}{a} = 0$).

   The triangle we are interested in is right-angled, with the legs $\frac{2}{a}$ and $2a$. Its area is $\frac{1}{2} \cdot \frac{2}{a} \cdot 2a = 2$, and does not depend on a.