Homework/Tutorial 6

A complete solution to questions 1 and 2 is worth 3 marks; for question 3 it is 1.4 marks; for the remaining questions it is 1.3 marks.

What this homework is about

You will learn how to apply the Intermediate Value Theorem, and how to check the continuity and compute the derivatives of the simplest functions.

Reminder

Intermediate Value Theorem. If a function f is continuous on the closed interval $[a, b]$, then it takes every real value between $f(a)$ and $f(b)$.

If a composition $f \circ g$ of two continuous functions f and g is defined on an interval (a, b), that it is itself continuous on (a, b).

If f is one-to-one and continuous on an interval or a ray, then its inverse f^{-1} is continuous on its domain.

Given a function f, the function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the **derivative of** f **with respect to** x. The domain of f' consists of all x for which the limit exists. The function f is said to be **differentiable at** x_0 if the limit above exists for $x = x_0$. When only the one-sided version $\lim\limits_{h \to 0^{\pm}}$ of the above limit is defined, we talk about **left-hand** and **right-hand derivatives** $f'_{\pm}(x)$.

The equation of the **tangent line** to the graph of a function f at the point x_0 (where f is differentiable) can be written as

$$y = f'(x_0)(x - x_0) + f(x_0).$$

A function $f(x)$ differentiable at $x = x_0$ is continuous at $x = x_0$, but the converse does not always hold.

A summary of differentiation rules:

$$(c)' = 0, \qquad (f \pm g)' = f' \pm g',$$
$$(x^r)' = rx^{r-1}, \qquad (fg)' = f'g + fg'.$$

Here c and r are any real numbers, and f and g are functions differentiable at the points of interest.

Questions

1. Consider the function $f(x) = \arcsin(|2x+1| - 2)$.
   (a) What is its natural domain?
   (b) Explain why f is continuous on its natural domain.
   (c) Show that the graph of f intersects the line $y = -\frac{x}{2}$ at least twice. (*Hint.* Use the Intermediate Value Theorem.)

2. Consider the functions

$$f(x) = \cos(\frac{1}{x}), \qquad g(x) = x\sqrt[3]{x}\cos(\frac{1}{x}).$$

   (a) What are their natural domains?
   (b) Compute the value of f at $x = \frac{1}{k\pi}$ for non-zero integer values of k. (The answer might depend on k.)
   (c) Does f have a limit at 0?
   (d) What is the discontinuity type of f at the point 0?
   (e) Is the function g even? odd?
   (f) What is the discontinuity type of g at the point 0?
   (g) Explain (briefly) why f and g are continuous on their natural domains.
   (h) Consider the function $h(x) = \begin{cases} g(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Show that it is differentiable at 0.

3. Recall that the *floor* $\lfloor x \rfloor$ of a real number x is defined as the greatest integer that is less than or equal to x. Compute the left-hand and the right-hand derivatives of the function $f(x) = \lfloor x \rfloor$. Plot the graph of f' and determine the type of its discontinuity points.

4. Find the equation of the tangent line to the graph of the function

$$f(x) = (x^2 + 1 + \frac{1}{(x-2)^3})(x^3 - \sqrt{x})$$

   at the point $x = 1$.

5. Compute the area of the triangle formed by the x-axis, the y-axis, and the tangent line to the hyperbola $y = \frac{1}{x}$ at the point $x = a$, where a is a positive real number. The answer might depend on a.