Homework/Tutorial 5

Please hand in your work at the end of the tutorial. Make sure you put your name and student ID number on what you hand in. Please write your work in an intelligible way!

A complete solution to question 1 is worth 3 marks; to questions 3 and 4 it is 2 marks; for the remaining ones it is 1 mark.

What this homework is about

You will compute finite at infinite limits both at finite points and at infinity; learn how to find all horizontal and vertical asymptotes of a function in a rigorous way; and practise in proving continuity and determining the type of discontinuity.

Reminder

A function f is said to have the limit $+\infty$ (or $-\infty$) at a, written as $\lim_{x \to \infty} f(x) = \pm \infty$, if the values f(x) increase (or decrease) without bound when x gets sufficiently close to a. One-sided infinite limits and infinite limits at infinity are defined similarly.

Whenever one of the conditions $\lim_{x\to a^{\pm}} f(x) = \pm \infty$ holds, the function f has a vertical asymptote x = a.

A function f is said to have the limit L at $\pm \infty$, written as $\lim_{x \to \pm \infty} f(x) = L$, if the values f(x) get as close as we like to L as x increases (or decreases) without bound.

Whenever one of the conditions $\lim_{x \to +\infty} f(x) = L$ holds, the function f has a horizontal asymptote y = L.

For rational functions, one has

$$\lim_{n \to \pm \infty} \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0} = \lim_{x \to \pm \infty} \frac{a_n x^n}{b_m x^m}$$

Here $a_n \neq 0, b_m \neq 0$.

A function f is called **continuous** at a if $\lim_{x \to a} f(x) = f(a)$ (and both the value and the limit at a are defined). A function is called continuous on an interval if it is continuous at every point of the interval (for the endpoints, one-sided continuity should be used).

Discontinuity types: removable, jump, infinite, oscillating, and mixed.

Rational and trigonometric functions are continuous on their domain. The sum, difference, product, ratio, and nth roots of continuous functions are continuous whenever defined.

Questions

- 1. Compute the following limits. In each case, determine the discontinuity type of the given function at the given point.
 - (a) $\lim_{x \to 1^{-}} f(x)$ and $\lim_{x \to 1^{+}} f(x)$, where $f(x) = \frac{x^2 3x + 2}{(x 1)^3}$; (b) $\lim_{x \to 0} \frac{\sin(2x)}{\sin(x)}$ (*hint:* use a trigonometric formula for $\sin(2x)$);
 - (c) $\lim_{x \to \frac{\pi}{2}^-} \tan(x)$ and $\lim_{x \to \frac{\pi}{2}^+} \tan(x)$.

(a)
$$f(x) = \frac{x^2 - 3x + 2}{(x - 1)^3} = \frac{(x - 1)(x - 2)}{(x - 1)^3} = \frac{x - 2}{(x - 1)^2}$$
 whenever $x \neq 1$. Since $(x - 1)^2 \ge 0$ for all x and $x - 2$ is close to $1 - 2 = -1 < 0$ for x close to 1, we have $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x - 2}{(x - 1)^2} = -\infty$, and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x - 2}{(x - 1)^2} = -\infty$. This is an infinite discontinuity.
(b) $\frac{\sin(2x)}{\sin(x)} = \frac{2\sin(x)\cos(x)}{\sin(x)} = 2\cos(x)$ whenever $\sin(x) \neq 0$. Since for x close to 0 and different from 0 the inequality $\sin(x) \neq 0$ does hold, we have $\lim_{x \to 0} \frac{\sin(2x)}{\sin(x)} = \lim_{x \to 0} (2\cos(x)) = 2$. At $x = 0$ the function is undefined since $\sin(0) = 0$. So, we have a removable continuity.
(c) $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and $\lim_{x \to \frac{\pi}{2}} \cos(x) = \cos(\frac{\pi}{2}) = 0$. For x close to $\frac{\pi}{2}$, we have:
• $\sin(x)$ is close to $\sin(\frac{\pi}{2}) = 1 > 0$;
• $\cos(x) > 0$ for $x < \frac{\pi}{2}$, and $\cos(x) < 0$ for $x > \frac{\pi}{2}$.
So, $\lim_{x \to \frac{\pi}{2}} \tan(x) = +\infty$ and $\lim_{x \to \frac{\pi}{2}} \tan(x) = -\infty$. This is an infinite discontinuity.

2. Compute $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$, where $f(x) = \sqrt{x^4 - x} - \sqrt{x^4 + 3x^2}$.

Solution. We know that for very big or very small x, the values of the polynomial $x^4 - x$ are close to its leading term x^4 , therefore $x^4 - x > 0$, and $\sqrt{x^4 - x}$ is defined. The same argument works for $\sqrt{x^4 + 3x^2}$. Moreover, $\sqrt{x^4 - x} + \sqrt{x^4 + 3x^2} > 0$ for very big or very small x, so we can freely multiply and divide by this expression. Now,

$$f(x) = \sqrt{x^4 - x} - \sqrt{x^4 + 3x^2}$$

$$= \frac{(\sqrt{x^4 - x} - \sqrt{x^4 + 3x^2})(\sqrt{x^4 - x} + \sqrt{x^4 + 3x^2})}{\sqrt{x^4 - x} + \sqrt{x^4 + 3x^2}}$$

$$= \frac{(x^4 - x) - (x^4 + 3x^2)}{\sqrt{x^4 - x} + \sqrt{x^4 + 3x^2}}$$

$$= \frac{-3x^2 - x}{\sqrt{x^4 - x} + \sqrt{x^4 + 3x^2}}$$

$$= \frac{-3x^2(1 + \frac{1}{3x})}{x^2(\sqrt{1 - \frac{1}{x^3}} + \sqrt{1 + \frac{3}{x^2}})}$$

$$= \frac{-3(1 + \frac{1}{3x})}{\sqrt{1 - \frac{1}{x^3}} + \sqrt{1 + \frac{3}{x^2}}} \xrightarrow{x \to \pm \infty} \frac{-3 \cdot 1}{1 + 1} = -1.5.$$

Here we used that $\sqrt{x^4} = |x^2| = x^2$, and that for very big or very small x one has $x \neq 0$.

- 3. Consider the function $f(x) = \frac{\sqrt{x^2 10x + 25}}{5 x}$.
 - (a) What is its natural domain?
 - (b) Compute $\lim_{x\to 5^-} f(x)$, $\lim_{x\to 5^+} f(x)$, $\lim_{x\to +\infty} f(x)$, $\lim_{x\to -\infty} f(x)$. (c) Explain why f is continuous on its natural domain.

 - (d) What is the discontinuity type of f at the point 5?
 - (e) Plot the graph of f.

Solution.
$$f(x) = \frac{\sqrt{x^2 - 10x + 25}}{5 - x} = \frac{\sqrt{(x - 5)^2}}{5 - x} = \frac{|5 - x|}{5 - x} = \begin{cases} 1, & x < 5; \\ -1, & x > 5. \end{cases}$$

We used that $|5 - x| = \begin{cases} 5 - x, & x < 5; \\ x - 5, & x > 5. \end{cases}$

- (a) The only possible problem here is the denominator, which is zero iff x = 5. So, the natural domain is $(-\infty, 5) \cup (5, +\infty)$.
- (b) $\lim_{x \to 5^{-}} f(x) = \lim_{x \to 5^{-}} 1 = 1.$ $\lim_{x \to 5^{+}} f(x) = \lim_{x \to 5^{+}} -1 = -1.$ $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} -1 = -1.$ $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 1 = 1.$
- (c) The natural domain of f consists of two open rays, $(-\infty, 5)$ and $(5, +\infty)$, on each of which f is constant, hence continuous.
- (d) Jump discontinuity.





4. Consider the function $f(x) = \frac{(x-1)\sqrt{x^2+2x+2}}{x^2+3x-4}$.

- (a) What is its natural domain?
- (b) Find its horizontal and vertical asymptotes.
- (c) Determine the type of all its discontinuities.
- (d) Sketch the graph of f.

Solution.
$$f(x) = \frac{(x-1)\sqrt{x^2+2x+2}}{x^2+3x-4} = \frac{(x-1)\sqrt{(x+1)^2+1}}{(x-1)(x+4)} = \frac{\sqrt{(x+1)^2+1}}{x+4}$$
 whenever $x \neq 1$.

- (a) Since $(x+1)^2 + 1 \ge 0 + 1 > 0$ for all x, $\sqrt{(x+1)^2 + 1}$ makes sense for all x. Further, the denominator $x^2 + 3x 4 = (x-1)(x+4)$ vanishes iff x = 1 or x = -4. So, the natural domain of f is $(-\infty, -4) \cup (-4, 1) \cup (1, +\infty)$.
- (b) We should check all the zeroes of the denominator.
 - x = -4: the denominator is zero and the numerator is not, so x = -4 is a vertical asymptote;
 - x = 1: both the numerator and the denominator are zero; after simplification by x - 1, we get $f(x) = \frac{\sqrt{(x+1)^2 + 1}}{x+4}$, and the denominator is no longer zero at x = 1; so, no vertical asymptote here.

To find horizontal asymptotes, we need to compute $\lim_{x \to \pm \infty} f(x)$.

$$\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{\sqrt{(x+1)^2 + 1}}{x+4}$$

$$= \lim_{x \to \pm \infty} \frac{\sqrt{x^2((1+\frac{1}{x})^2 + \frac{1}{x^2})}}{x(1+\frac{4}{x})}$$

$$= \lim_{x \to \pm \infty} \left(\frac{|x|}{x} \cdot \frac{\sqrt{(1+\frac{1}{x})^2 + \frac{1}{x^2}}}{1+\frac{4}{x}}\right)$$

$$= \lim_{x \to \pm \infty} \frac{|x|}{x} \frac{\sqrt{(1+0)^2 + 0}}{1+0} = \lim_{x \to \pm \infty} \frac{|x|}{x}.$$
So, $\lim_{x \to +\infty} f(x) = \lim_{x \to \pm \infty} 1 = 1$ and $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -1 = -1$. Therefore, f has two horizontal asymptotes: $y = 1$ and $y = -1$.
We used that $|x| = \begin{cases} x, & x > 0; \\ -x, & x < 0. \end{cases}$
(c) Infinite discontinuity at $x = -4$ (since the denominator is zero and the numerator is not); removable discontinuity at $x = 1$ (since $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\sqrt{(x+1)^2 + 1}}{x+4} = \frac{\sqrt{(1+1)^2 + 1}}{1+4} = \frac{\sqrt{5}}{5}$).

x

5. Suppose that a function g satisfies $\lim_{y \to a} g(y) = 3$. Compute $\lim_{y \to a} \frac{g(y) + 1}{g(y) - 2}$.

Solution. The function we need to study can be presented as a composition $f \circ g$, where $f(x) = \frac{x+1}{x-2}$. We know that $\lim_{y \to a} g(y) = 3$, and we compute $\lim_{x \to 3} f(x) = \frac{3+1}{3-2} = 4$. By a theorem seen in class, we conclude $\lim_{y \to a} \frac{g(y)+1}{g(y)-2} = \lim_{y \to a} f(g(y)) = \lim_{x \to 3} f(x) = 4$.

6. Consider the function P(t) plotted below. It shows how the size P(t) of an ecological system depends on time t, according to the logistic growth model. From the graph,

determine $\lim_{t \to +\infty} P(t)$.



Solution. We see that as t grows, the values of P(t) approach N. (In other words, y = N is the horizontal asymptote of the graph at $+\infty$.) This means that $\lim_{t \to +\infty} P(t) = N$.