Homework/Tutorial 5

Please hand in your work at the end of the tutorial. Make sure you put your name and student ID number on what you hand in. Please write your work in an intelligible way!

A complete solution to question 1 is worth 3 marks; to questions 3 and 4 is worth 2 marks; for the remaining ones it is 1 mark.

What this homework is about

You will compute finite at infinite limits both at finite points and at infinity; learn how to find all horizontal and vertical asymptotes of a function in a rigorous way; and practise in proving continuity and determining the type of discontinuity.

Reminder

A function f is said to **have the limit $+\infty$ (or $-\infty$) at** a, written as $\lim_{x\to a} f(x) = \pm\infty$, if the values $f(x)$ increase (or decrease) without bound when x gets sufficiently close to a. One-sided infinite limits and infinite limits at infinity are defined similarly.

Whenever one of the conditions $\lim_{x\to a^\pm} f(x) = \pm\infty$ holds, the function f has a **vertical asymptote** $x = a$.

A function f is said to **have the limit** L **at** $\pm\infty$, written as $\lim_{x\to\pm\infty} f(x) = L$, if the values $f(x)$ get as close as we like to L as x increases (or decreases) without bound.

Whenever one of the conditions $\lim_{x\to\pm\infty} f(x) = L$ holds, the function f has a **horizontal asymptote** $y = L$.

For rational functions, one has

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}.$$

Here $a_n \neq 0$, $b_m \neq 0$.

A function f is called **continuous** at a if $\lim_{x\to a} f(x) = f(a)$ (and both the value and the limit at a are defined). A function is called continuous on an interval if it is continuous at every point of the interval (for the endpoints, one-sided continuity should be used).

Discontinuity types: removable, jump, infinite, oscillating, and mixed.

Rational and trigonometric functions are continuous on their domain. The sum, difference, product, ratio, and nth roots of continuous functions are continuous whenever defined.

Questions

1. Compute the following limits. In each case, determine the discontinuity type of the given function at the given point.
   (a) $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$, where $f(x) = \dfrac{x^2 - 3x + 2}{(x-1)^3}$;
   (b) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(x)}$ (*hint:* use a trigonometric formula for $\sin(2x)$);
   (c) $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$ and $\lim_{x\to \frac{\pi}{2}^+} \tan(x)$.
2. Compute $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, where $f(x) = \sqrt{x^4 - x} - \sqrt{x^4 + 3x^2}$.

3. Consider the function $f(x) = \dfrac{\sqrt{x^2 - 10x + 25}}{5 - x}$.
   (a) What is its natural domain?
   (b) Compute $\lim_{x \to 5^-} f(x)$, $\lim_{x \to 5^+} f(x)$, $\lim_{x \to +\infty} f(x)$, $\lim_{x \to -\infty} f(x)$.
   (c) Explain why f is continuous on its natural domain.
   (d) What is the discontinuity type of f at the point 5?
   (e) Plot the graph of f.

4. Consider the function $f(x) = \dfrac{(x-1)\sqrt{x^2 + 2x + 2}}{x^2 + 3x - 4}$.
   (a) What is its natural domain?
   (b) Find its horizontal and vertical asymptotes.
   (c) Determine the type of all its discontinuities.
   (d) Sketch the graph of f.

5. Suppose that a function g satisfies $\lim_{y \to a} g(y) = 3$. Compute $\lim_{y \to a} \dfrac{g(y) + 1}{g(y) - 2}$.

6. Consider the function $P(t)$ plotted below. It shows how the size $P(t)$ of an ecological system depends on time t, according to the logistic growth model. From the graph, determine $\lim_{t \to +\infty} P(t)$.