Homework/Tutorial 4

Please hand in your work at the end of the tutorial. Make sure you put your name and student ID number on what you hand in. Please write your work in an intelligible way!

A complete solution to question 1 is worth 3 marks, to question 4 is worth 1 mark, and for the remaining questions it is 2 marks.

What this homework is about

In exercises 1-3 you will continue working with inverse functions, in particular with inverse trigonometric functions. In exercises 4-5 you will start computing limits.

Reminder

Trigonometric functions are periodic, and so do not have inverses, but their restrictions to certain intervals do. Here is a summary on classical inverse trigonometric functions:

function	domain	range	symmetries
arcsin	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	odd
arccos	$[-1,1]$	$[0,\pi]$	–
arctan	$\mathbb{R}$	$(-\frac{\pi}{2},\frac{\pi}{2})$	odd
arccot	$\mathbb{R}$	$(0,\pi)$	–

A function f **has the limit** L **at the point** a if the values $f(x)$ get as close as we wish to L when x gets sufficiently close to a. This is written as $\lim\limits_{x\to a} f(x) = L$, or $f(x) \underset{x\to a}{\to} L$.

If the condition above is true only for $x > a$ (or $x < a$), one talks about **one-sided limits**, and writes $x \to a^+$ (or $x \to a^-$) in the notations above. Two- and one-sided limits are related as follows:

$$\lim_{x\to a} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Questions

1. (a) Determine all real c and d for which the function
$$f(x) = x^2 + cx + d, \qquad x \geq 0$$
admits an inverse.
   (b) Write down this inverse when it exists, and find its domain and range.
   (c) Plot the graph of the inverse function for $c = 2$ and $d = 2$.
   (d) When d varies, what happens to the graphs of f and f^{-1}?

   Solution.
   (a) By completing the square, rewrite f as $f(x) = (x+\frac{c}{2})^2 - \frac{c^2}{4} + d$, $x \geq 0$. For $x \geq -\frac{c}{2}$, the function is increasing.
   - If $c \geq 0$, then $-\frac{c}{2} \leq 0$, so f is also increasing on $[0,+\infty)$, and is thus invertible.
   - If $c < 0$, then f is not injective on $[0,+\infty)$: for instance, 0 and $-c$ are different numbers from the domain of f, and $f(0) = f(-c)$ since $(0+\frac{c}{2})^2 = (\frac{c}{2})^2 = (-\frac{c}{2})^2 = (c-\frac{c}{2})^2$. So, in this case f is not invertible.

(b) We have seen that f^{-1} exists only for $c \geq 0$. Let us compute f^{-1} in this case:

$$\begin{aligned} & y = (x+\frac{c}{2})^2 - \frac{c^2}{4} + d \\ \iff \quad & y + \frac{c^2}{4} - d = (x+\frac{c}{2})^2 \\ \implies \quad & x + \frac{c}{2} = \sqrt{y + \frac{c^2}{4} - d} \\ \iff \quad & x = \sqrt{y + \frac{c^2}{4} - d} - \frac{c}{2}. \end{aligned}$$

So, $f^{-1}(x) = \sqrt{x + \frac{c^2}{4} - d} - \frac{c}{2}$. It remains to determine its domain, i.e., the range of f. Since f is increasing and becomes arbitrarily large for large x, its range is $[f(0), +\infty)$, which is $[d, +\infty)$.

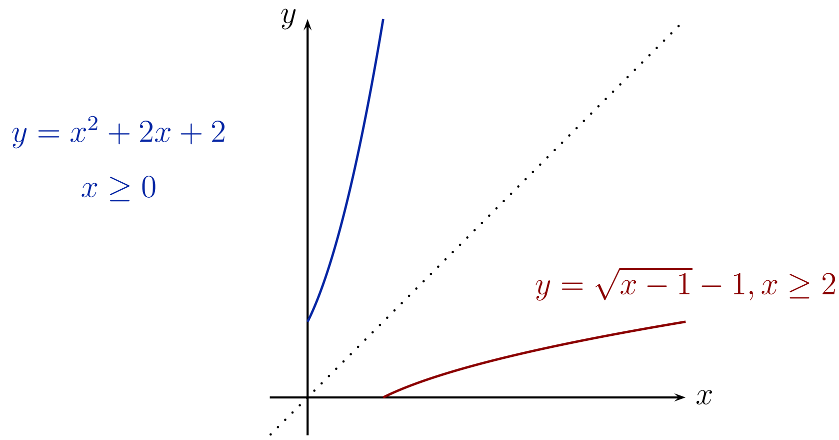
Answer: $f^{-1}(x) = \sqrt{x + \frac{c^2}{4} - d} - \frac{c}{2}$, $x \geq d$.

The range of f^{-1} is the domain of f, which is $[0, +\infty)$.

(c) The graph of f is a part of the shifted parabola. As we saw above, it increases from d to $+\infty$. The graph of

$$f^{-1}(x) = \sqrt{x + \frac{2^2}{4} - 2} - \frac{2}{2} = \sqrt{x-1} - 1, \qquad x \geq 2$$

is obtained by taking reflection about the line $y = x$.



(d) When d varies, the graphs of f shifts vertically, and the graph of f^{-1} shifts horizontally, since for $f^{-1}(x)$ changing d to $d + e$ corresponds to changing x to $x - e$.

2. Compute
   (a) $\tan(\arctan(3))$;
   (b) $\sin(\arcsin(-2))$;
   (c) $\arccos(\cos(-\frac{\pi}{3}))$;
   (d) $\arctan(\tan(\frac{5\pi}{8}))$.

Solution.

(a) $\tan(\arctan(3)) = 3$, since, by the definition of the inverse function, $\tan(\arctan(x)) = x$ for all x from the domain of arctan, which is $\mathbb{R}$.

(b) $\sin(\arcsin(-2))$ is not defined, since -2 lies outside the domain of arcsin, which is $[-1, 1]$.

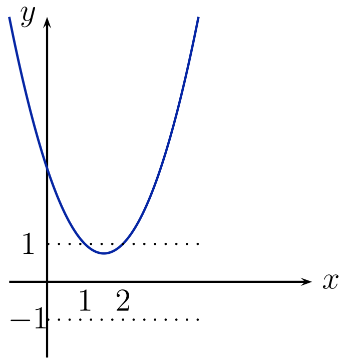
(c) $\arccos(\cos(-\frac{\pi}{3})) = \arccos(\cos(\frac{\pi}{3})) = \frac{\pi}{3}$. We used that cos is an even function, and that $\arccos(\cos(x)) = x$ for all $x \in [0, \pi]$, which includes $\frac{\pi}{3}$ (but not $-\frac{\pi}{3}$!!!).

(d) $\arctan(\tan(\frac{5\pi}{8})) = \arctan(\tan(\frac{5\pi}{8} - \pi)) = \arctan(\tan(-\frac{3\pi}{8})) = -\frac{3\pi}{8}$. We used that tan is π-periodic, and that $\arctan(\tan(x)) = x$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which includes $-\frac{3\pi}{8}$ (but not $\frac{5\pi}{8}$!!!).

3. Determine the natural domain of the function

$$f(x) = \arccos(x^2 - 3x + 3) - \frac{x}{\arctan(x - \frac{\pi}{2})}.$$

Solution. First, we need to check that $x^2 - 3x + 3$ is in the domain of arccos, which is $[-1, 1]$. The equation $x^2 - 3x + 3 = 1$ rewrites as $(x-2)(x-1) = 0$, and has two solutions: $x = 1$ and $x = 2$. The equation $x^2 - 3x + 3 = -1$ has no real solutions, since the determinant of $x^2 - 3x + 4$ is $(-3)^2 - 4 \cdot 4 = -7 < 0$. So, $x^2 - 3x + 3$ lies in $[-1, 1]$ when $x \in [1, 2]$.



Further, arctan is defined everywhere, so $\frac{x}{\arctan(x - \frac{\pi}{2})}$ is defined unless the denominator $\arctan(x - \frac{\pi}{2})$ is zero, which happens when $x = \frac{\pi}{2} \approx 1.57$, which is between 1 and 2.

Answer: The natural domain of the whole function is $[1, \frac{\pi}{2}) \cup (\frac{\pi}{2}, 2]$.

4. Compute $\lim_{x \to -2} \frac{x^2-4}{x+2}$.

Solution. $\lim_{x \to -2} \frac{x^2-4}{x+2} = \lim_{x \to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x \to -2} x - 2 = -2 - 2 = -4$. We could simplify by $x + 2$ since it vanishes only for $x = -2$, and the value at -2 is irrelevant for computing the limit at -2.

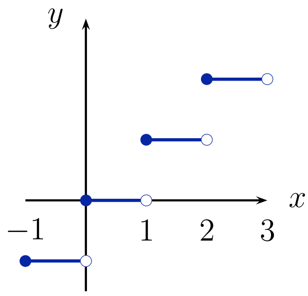
5. The *floor* $\lfloor x \rfloor$ of a real number x is defined as the greatest integer that is less than or equal to x. For instance, $\lfloor 0.7 \rfloor = 0$, $\lfloor \pi \rfloor = 3$, $\lfloor -0.7 \rfloor = -1$, $\lfloor -4 \rfloor = -4$.

(a) Plot the graph of the function $f(x) = \lfloor x \rfloor$.

(b) Compute its one-sided and two-sided limits at $x = 1$, if they exist.

Solution.

(a) By definition, for all integers k, one has $\lfloor x \rfloor = k$ for all $x \in [k, k+1)$, so our function is constant on each half-closed interval $[k, k+1)$, and has a staircase shape:



(b) Since $\lfloor x \rfloor = 0$ for all $x \in [0, 1)$, we get $\lim_{x\to 1^-} \lfloor x \rfloor = 0$. Similarly, $\lfloor x \rfloor = 1$ for all $x \in [1, 2)$, so $\lim_{x\to 1^+} \lfloor x \rfloor = 1$. Since the one-sided limits are distinct at 1, f has no two-sided limit at 1.