Homework/Tutorial 3

Please hand in your work at the end of the tutorial. Make sure you put your name and student ID number on what you hand in. Please write your work in an intelligible way!

A complete solution to questions 1, 2, 3 is worth 2, 3, 5 marks respectively.

What this homework is about

You'll work with basic functions and their graphs. You'll learn how to determine the symmetries of functions and graphs, and relate them. You'll also practise in constructing inverse functions, and telling whether they exist.

Reminder

A function f is called **even** (resp., **odd**) if for all x in the domain of f, $-x$ is also in the domain of f, and $f(-x) = f(x)$ (resp., $f(-x) = -f(x)$). In this case, the graph of f is symmetric with respect to the y-axis (resp., with respect to the origin).

A function f is called **periodic with period** T if for all x in the domain of f, $x + T$ is also in the domain of f, and $f(x + T) = f(x)$. In this case, the graph of f remains unchanged when shifted by T units to the left or to the right.

Suppose that for a function f there exists a function g such that

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g,$$
$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

Then g is called the **inverse** of f, and is denoted by f^{-1}. The graph of f^{-1} is the reflection of the graph of f with respect to the line $y = x$. A function has an inverse if and only if it is **injective**, that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

A **vertical/horizontal asymptote** of a graph is a vertical/horizontal line that the graph closely approximates.

Questions

1. Let f and g be two functions such that their composition $f \circ g$ is defined for all x in the domain of g. Suppose that g is periodic with period T. Show that $f \circ g$ is also periodic with period T.
   Example. Prove that the function
   $$h(x) = \frac{\sin(2x + \frac{\pi}{3})^2 + 5}{2 - \sin(2x + \frac{\pi}{3})}$$
   is periodic, with a period which you will determine.

   Solution. By assumption, f is defined at the point $g(x)$ for any x in the domain of g. Therefore, the domain of $f \circ g$ is simply the domain of g. Since g is T-periodic, given any x in the domain of g, the point $x + T$ is also in the domain of g, and $g(x + T) = g(x)$. Then $x + T$ is in the domain of $f \circ g$ as well, and $f(g(x + T)) = f(g(x))$. This means that $f \circ g$ is T-periodic.
   The function h from the example can be decomposed as $h = f \circ g$, where $f(x) = \frac{x^2+5}{2-x}$, and $g(x) = \sin(2x + \frac{\pi}{3})$. The function $g(x)$ is defined for all real x, and its range is $[-1, 1]$. It is periodic with period $\frac{2\pi}{2} = \pi$. The function $f(x)$ is defined whenever $x \neq 2$. Since g never takes the value 2, $f \circ g$ is defined for all x in the domain of g. From what we proved above, we can conclude that $f \circ g$ is periodic with period π.

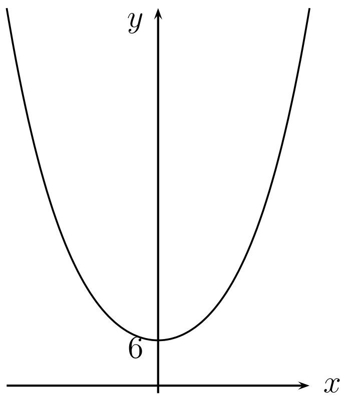
2. (a) What is the degree of the polynomial $x^4 + 7x^2 + 6$?
   (b) Present the function $f(x) = x^4 + 7x^2 + 6$ as a composition of quadratic polynomial functions.
   (c) Find the domain and the range of f.
   (d) Is f odd? even?
   (e) Does it have an inverse?
   (f) Sketch a graph of f. (I don't ask for precision, I just want to see the symmetries if there are any, the parts where f is increasing/decreasing, and the general shape of the graph.)
   (g) The graph of what simple function does it approach for large $|x|$?

Solution.

(a) 4.
(b) $f = g \circ h$, $g(x) = x^2 + 7x + 6$, $h(x) = x^2$.
(c) Domain: $\mathbb{R}$ (since the polynomial $f(x)$ is defined for all real x). Range: $[6, +\infty)$. Indeed, we need to understand what values $g(x)$ takes for x from the range of h, which is $[0, +\infty)$. The function $g(x)$ increases when x changes from 0 to $+\infty$ (since both x^2 and x increase on $[0, +\infty)$). So, its values on $[0, +\infty)$ change between $g(0) = 6$ and $+\infty$ (when x becomes arbitrarily large, so does $g(x)$).
(d) All conditions on the domain are automatic here, since the domain is the whole real line $\mathbb{R}$. Let us look at the values. Since

$$f(-x) = (-x)^4 + 7(-x)^2 + 6 = x^4 + 7x^2 + 6 = f(x)$$

for all x, f is even. Since $f(-1) = 14$ is different from $-f(1) = -14$, f is not odd.
(e) No, since it is even. You can also argue that $f(-1) = 14 = f(1)$, so $f^{-1}(14)$ needs to be 1 and -1 at the same time.
(f) We know that f is even, so we should work only with $x \geq 0$, and then take symmetry about the y-axis. When x changes from 0 to $+\infty$, so does $h(x) = x^2$, and, as seen above, $f(x) = g(h(x))$ then increases from 6 to $+\infty$. The graph then looks as follows (we scaled in the y-direction, so that the picture takes less place):

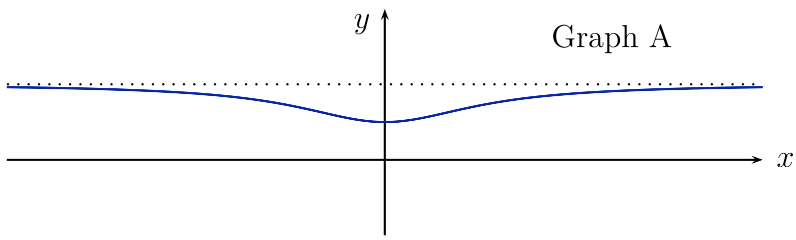


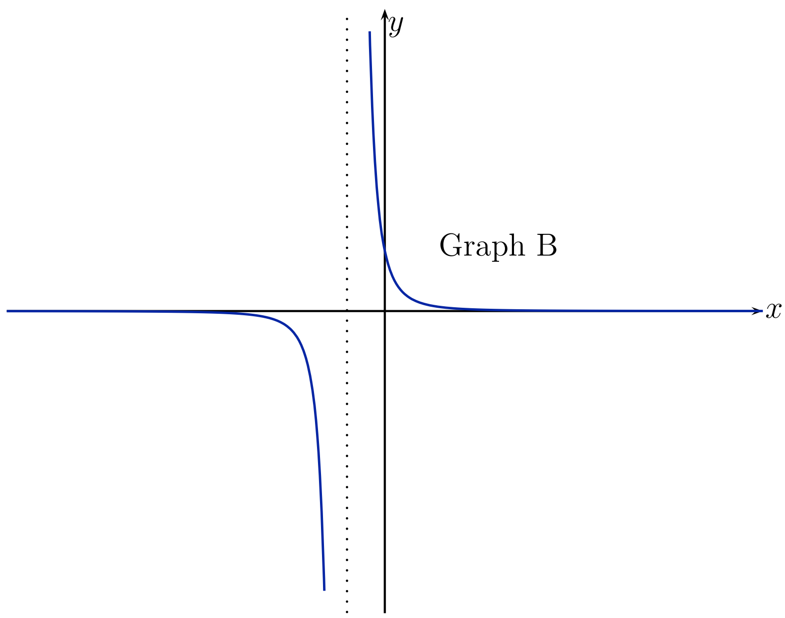
(g) The graph of x^4, which is its leading term (seen in class).

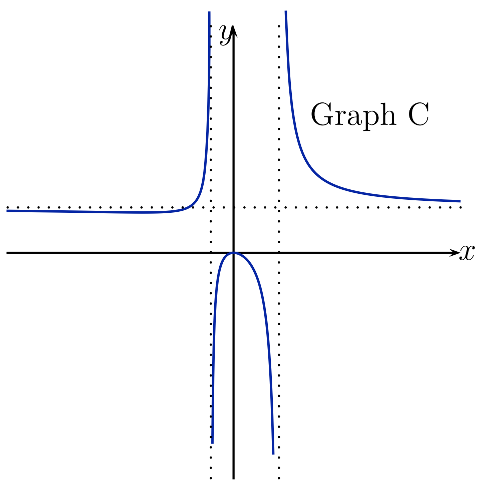
3. (a) The following three graphs A, B, and C correspond (in some order) to the functions

$$f(x) = \frac{x^2 + 1}{x^2 + 2}, \qquad g(x) = \frac{x^2}{x^2 - x - 2}, \qquad h(x) = \frac{4}{(x+1)^3}.$$

   Match the graphs with the functions, and explain your reasoning.
   (b) Determine the natural domains of the three functions.
   (c) Tell if any of these functions is even/odd, and how it can be seen from their graphs.
   (d) Give equations for horizontal and vertical asymptotes of all the graphs.
   (e) Do these functions have inverses? If yes, determine them.







Solution.

(a) The only even function is f, and the only vertically symmetric graph is A, so A is the graph of f. Further, h has only one vertical asymptote, since the denominator $(x+1)^3$ is zero only at $x = -1$. Now, as we see two vertical asymptotes in graph C, it cannot be the graph of h. The only remaining possibility is that B is the graph of h, and C that of g.

(b) Since the denominator $x^2 + 2$ of f is positive for all real x, the natural domain of f is the whole real line $\mathbb{R}$.
The denominator of g factorises as $x^2 - x - 2 = (x-2)(x+1)$, which is zero when $x = -1$ or $x = 2$. So, its natural domain is $(-\infty, -1) \cup (-1, 2) \cup (2, +\infty)$.

The denominator $(x+1)^3$ of h is zero only at $x = -1$, so its natural domain is $(-\infty, -1) \cup (-1, +\infty)$.

(c) Since f is defined for all real x and $f(-x) = f(x)$ for all x, the function is even, and its graph is vertically symmetric. It is not odd, since for instance $f(-1) = \frac{2}{3}$, which is different from $-f(1) = -\frac{2}{3}$.
Neither g nor h have a domain symmetric about 0, so they are neither even nor odd. Their graphs have no symmetry about the y-axis, nor about the origin.

(d)

function	horizontal asymptotes	vertical asymptotes
f	–	$y = 1$
g	$x = -1$ or $x = 2$	$y = 1$
h	$x = -1$	$y = 0$

All our functions are rational functions. So, for vertical asymptotes, we need to look at the zeros of the denominators, and to check that they are not also the zeros of the numerator, to be sure that near these points the absolute value of the function takes arbitrarily large values.
For horizontal asymptotes, we need to choose the leading terms of the numerator and the denominator, and study their ratio. For f and g, the ratio is $\frac{x^2}{x^2} = 1$, so both functions have a vertical asymptote $y = 1$. For h, the ratio is $\frac{4}{x^3}$, which becomes close to 0 for large values of $|x|$. So, h has a vertical asymptote $y = 0$.

(e) The functions f and g do not have inverses, because there are not injective. Indeed, f is even, and for g you can see from its graph that many horizontal lines hit the graph in at least two points.
To show that h has an inverse, we will simply compute this inverse, using the algorithm from lectures. First, we need to express x from $y = \frac{4}{(x+1)^3}$:

$$
\begin{aligned}
& y = \frac{4}{(x+1)^3}, \\
\iff \quad & y = \frac{4}{(x+1)^3} = 0, \text{ or } \frac{1}{y} = \frac{(x+1)^3}{4}, \\
\iff \quad & \frac{4}{y} = (x+1)^3, \\
\iff \quad & \sqrt[3]{\frac{4}{y}} = x + 1, \\
\iff \quad & \sqrt[3]{\frac{4}{y}} - 1 = x.
\end{aligned}
$$

We discarded the option $y = \frac{4}{(x+1)^3} = 0$, since $\frac{4}{(x+1)^3}$ cannot be zero.
Next, we need to determine the domain of h^{-1}, which is the range of h. Since h is a horizontal shift of the function $\frac{4}{x^3}$, is has the same range, which is $(-\infty, 0) \cup (0, +\infty)$.
So, the inverse of h is the function $h^{-1}(x) = \sqrt[3]{\frac{4}{x}} - 1$, $x \neq 0$.