Homework/Tutorial 3

Please hand in your work at the end of the tutorial. Make sure you put your name and student ID number on what you hand in. Please write your work in an intelligible way!

A complete solution to questions 1, 2, 3 is worth 2, 3, 5 marks respectively.

What this homework is about

You'll work with basic functions and their graphs. You'll learn how to determine the symmetries of functions and graphs, and relate them. You'll also practise in constructing inverse functions, and telling whether they exist.

Reminder

A function f is called **even** (resp., **odd**) if for all x in the domain of f, -x is also in the domain of f, and f(-x) = f(x) (resp., f(-x) = -f(x)). In this case, the graph of f is symmetric with respect to the y-axis (resp., with respect to the origin).

A function f is called **periodic with period** T if for all x in the domain of f, x + T is also in the domain of f, and f(x + T) = f(x). In this case, the graph of f remains unchanged when shifted by T units to the left or to the right.

Suppose that for a function f there exists a function g such that

f(g(x)) = x for all x in the domain of g,

g(f(x)) = x for all x in the domain of f.

Then g is called the **inverse** of f, and is denoted by f^{-1} . The graph of f^{-1} is the reflection of the graph of f with respect to the line y = x. A function has an inverse if and only if it is **injective**, that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

A vertical/horizontal asymptote of a graph is a vertical/horizontal line that the graph closely approximates.

Questions

1. Let f and g be two functions such that their composition $f \circ g$ is defined for all x in the domain of g. Suppose that g is periodic with period T. Show that $f \circ g$ is also periodic with period T.

Example. Prove that the function

$$h(x) = \frac{\sin(2x + \frac{\pi}{3})^2 + 5}{2 - \sin(2x + \frac{\pi}{3})}$$

is periodic, with a period which you will determine.

- 2. (a) What is the degree of the polynomial $x^4 + 7x^2 + 6$?
 - (b) Present the function $f(x) = x^4 + 7x^2 + 6$ as a composition of quadratic polynomial functions.
 - (c) Find the domain and the range of f.
 - (d) Is f odd? even?
 - (e) Does it have an inverse?
 - (f) Sketch a graph of f. (I don't ask for precision, I just want to see the symmetries if there are any, the parts where f is increasing/decreasing, and the general shape of the graph.)
 - (g) The graph of what simple function does it approach for large |x|?

3. (a) The following three graphs A, B, and C correspond (in some order) to the functions

$$f(x) = \frac{x^2 + 1}{x^2 + 2},$$
 $g(x) = \frac{x^2}{x^2 - x - 2},$ $h(x) = \frac{4}{(x+1)^3}$

Match the graphs with the functions, and explain your reasoning.

- (b) Determine the natural domains of the three functions.
- (c) Tell if any of these functions is even/odd, and how it can be seen from their graphs.
- (d) Give equations for horizontal and vertical asymptotes of all the graphs.
- (e) Do these functions have inverses? If yes, determine them.

