Homework 3: Representations of S_n and A_n.

Instructions. Try to give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those.

Exercise 1. Our aim is to recover the irrep $V_{n-2,1,1}$ of the symmetric group S_n as the alternating square $\Lambda^2(V^{st})$, and the irrep $V_{n-2,2}$ as a direct summand of the symmetric square $S^2(V^{st})$. First, take positive integers $p \geq q \geq 1$. Consider the two-part partition (p, q) of $p+q$.

1. Prove that the following S_{p+q}-representations are isomorphic:
$$\mathbb{C}M_{p,q} \cong V_{p,q} \oplus \mathbb{C}M_{p+1,q-1},$$
using the Frobenius formula for the character of V_λ and its analogue for $\mathbb{C}M_\lambda$.

   Solution. By the Frobenius formula, $\chi^{V_{p,q}}(\sigma)$ is the coefficient of the monomial $x_1^{p+1}x_2^q$ in the polynomial $(x_1 - x_2)P_{\lambda'}(x_1, x_2)$, where
$$P_{\lambda'}(x_1, x_2) = (x_1^{\lambda'_1} + x_2^{\lambda'_1}) \cdots (x_1^{\lambda'_{k'}} + x_2^{\lambda'_{k'}}),$$
and the permutation σ is of cycle type $\lambda' = (\lambda'_1, \ldots, \lambda'_{k'})$. Writing $(x_1 - x_2)P_{\lambda'}(x_1, x_2)$ as $x_1 P_{\lambda'}(x_1, x_2) - x_2 P_{\lambda'}(x_1, x_2)$, one sees that the coefficient of $x_1^{p+1}x_2^q$ in $(x_1 - x_2)P_{\lambda'}(x_1, x_2)$ is the coefficient of $x_1^p x_2^q$ in $P_{\lambda'}(x_1, x_2)$, minus the coefficient of $x_1^{p+1}x_2^{q-1}$ in $P_{\lambda'}(x_1, x_2)$. Recalling the formula for the character of $\mathbb{C}M_\lambda$, one concludes
$$\chi^{V_{p,q}}(\sigma) = \chi^{\mathbb{C}M_{p,q}}(\sigma) - \chi^{\mathbb{C}M_{p+1,q-1}}(\sigma).$$
The representations $\mathbb{C}M_{p,q}$ and $V_{p,q} \oplus \mathbb{C}M_{p+1,q-1}$ have the same character, and are therefore isomorphic.

2. Deduce from this the degree of $V_{p,q}$. Compare with the result predicted by the Hook length formula.

   Solution. The set of Young tabloids M_λ is the set of Young tableaux of shape $\lambda \vdash n$, considered up to row permutation. Such a tabloid is uniquely determined by which λ_1 numbers from $\{1, 2, \ldots, n\}$ are in the first row, with λ_2 numbers are in the second row, etc. The cardinal of M_λ is thus
$$\binom{n}{\lambda_1, \lambda_2, \ldots} = \frac{n!}{\lambda_1! \lambda_2! \cdots}.$$
Therefore,
$$\begin{aligned}\dim_{\mathbb{C}}(V_{p,q}) &= \dim_{\mathbb{C}}(\mathbb{C}M_{p,q}) - \dim_{\mathbb{C}}(\mathbb{C}M_{p+1,q-1}) = \#M_{p,q} - \#M_{p+1,q-1} \\ &= \frac{(p+q)!}{p!q!} - \frac{(p+q)!}{(p+1)!(q-1)!} = \frac{(p+q)!}{(p+1)!q!}(p+1-q).\end{aligned}$$
Further, the hook length of all the cells in $D_{p,q}$ is computed as follows:

$p{+}1$	p	$\ldots$	$p{-}q{+}2$	$p{-}q$	$p{-}q{-}1$	$\ldots$	1
q	$q{-}1$	$\ldots$	1				

The Hook length formula then yields

$$\dim_{\mathbb{C}}(V_{p,q}) = \frac{(p+q)!}{(p+1)\cdots(p-q+2)(p-q)!q!} = \frac{(p+q)!}{(p+1)!q!}(p-q+1).$$

The two methods give the same answer.

3. Decompose the S_{p+q}-representation $\mathbb{C}M_{p,q}$ into irreducibles.

   Solution. Iterating the formula from Q1, and using $V_{p+q,0} = V_{p+q} \cong V^{tr}$, one gets

   $$\mathbb{C}M_{p,q} \cong V_{p,q} \oplus V_{p+1,q-1} \oplus \cdots \oplus V_{p+q-1,1} \oplus V^{tr}.$$

   Since the Specht representations V_λ are all irreducible, this is the desired decomposition.

Now, take an integer $n \geq 4$.

4. As a particular case of the previous question, show that the following S_n-representations are isomorphic:

   $$\mathbb{C}M_{n-2,2} \cong V_{n-2,2} \oplus V^{st} \oplus V^{tr}.$$

   Solution. This decomposition is obtained from Q3, using $V_{p+q-1,1} \cong V^{st}$.

5. Recall the permutation representation $V^p = \oplus_{i=1}^n \mathbb{C}e_i$ of S_n, with $\sigma \cdot e_i = e_{\sigma(i)}$. Verify that the following S_n-representations are isomorphic:

   $$V^p \otimes V^p \cong V^{st} \otimes V^{st} \oplus 2V^{st} \oplus V^{tr},$$
   $$\Lambda^2(V^p) \cong \Lambda^2(V^{st}) \oplus V^{st}.$$

   Solution. The standard representation V^{st} was defined as the complement of a degree 1 trivial sub-representation in V^p, so $V^p \cong V^{st} \oplus V^{tr}$. Using the arithmetic properties of the operations $\oplus$ and $\otimes$ on $\mathrm{Rep}(S_n)$, we get

   $$\begin{aligned} V^p \otimes V^p &\cong (V^{st} \oplus V^{tr}) \otimes (V^{st} \oplus V^{tr}) \\ &\cong V^{st} \otimes V^{st} \oplus V^{st} \otimes V^{tr} \oplus V^{tr} \otimes V^{st} \oplus V^{tr} \otimes V^{tr} \cong V^{st} \otimes V^{st} \oplus 2V^{st} \oplus V^{tr}. \end{aligned}$$

   Further, let $b_1, \ldots, b_{n-1}$ be a basis of V^{st}, completed to a basis $b_0, b_1, \ldots, b_{n-1}$ of V^p in such a way that b_0 spans a sub-representation $\cong V^{tr}$. A basis of $\Lambda^2(V^p)$ is then given by $\{b_i \otimes b_j - b_j \otimes b_i \,|\, 0 \leq i < j < n\}$. Similarly, a basis of $\Lambda^2(V^{st})$ is given by $\{b_i \otimes b_j - b_j \otimes b_i \,|\, 1 \leq i < j < n\}$. It remains to show that the vectors $\{f(b_j) := b_0 \otimes b_j - b_j \otimes b_0 \,|\, 1 \leq j < n\}$ span a sub-representation W of $\Lambda^2(V^p)$ isomorphic to V^{st}. For any $\sigma \in S_n$, one has

   $$\sigma \cdot f(b_j) = \sigma \cdot (b_0 \otimes b_j - b_j \otimes b_0) = \sigma \cdot b_0 \otimes \sigma \cdot b_j - \sigma \cdot b_j \otimes \sigma \cdot b_0 = f(\sigma \cdot b_j).$$

   We used the consequence $\sigma \cdot b_0 = b_0$ of $\mathbb{C}b_0 \cong V^{tr}$. Hence W is indeed a sub-representation, and the linear map $V^{st} \to W$ defined on generators as $b_j \mapsto f(b_j)$ is an S_n-rep. isomorphism.

6. Using the interpretation of $M_{n-2,2}$ in terms of Young tabloids, prove

   $$S^2(V^p) \cong \mathbb{C}M_{n-2,2} \oplus V^p,$$

   and decompose the symmetric square $S^2(V^p)$ into irreducibles.

   Solution. A basis of $S^2(V^p)$ is given by $\{e_{i,j} := e_i \otimes e_j + e_j \otimes e_i \,|\, 1 \leq i \leq j \leq n\}$. As was done in the previous question, one shows that, for any $\sigma \in S_n$, $\sigma \cdot e_{i,i} = e_{\sigma(i),\sigma(i)}$ for all i, and $\sigma \cdot e_{i,j} = e_{\sigma(i),\sigma(j)}$ or $e_{\sigma(j),\sigma(i)}$ for all $i < j$, according to which of $\sigma(i)$ and $\sigma(j)$ is smaller. Thus the $e_{i,i}$ span a sub-representation of $S^2(V^p)$ isomorphic to V^p: the isomorphism is established by sending $e_{i,i}$ to e_i. Similarly, the $e_{i,j}$ with $i < j$ span a sub-representation isomorphic to $\mathbb{C}M_{n-2,2}$. The latter isomorphism is established by sending $e_{i,j}$ to the Young tabloid of shape $(n-2,2)$, with the entries i,j in the second row, and all the remaining entries from $\{1,2,\dots,n\}$ in the first row. Recall that Young tabloids are considered up to row permutation, so we indeed have a bijection $\{e_{i,j} \,|\, 1 \le i < j \le n\} \to M_{n-2,2}$. Further, σ acts on a Young tabloid by acting on the numbers in each cell. Hence our bijection is compatible with the S_n-action, and thus defines an isomorphism of S_n-reps $\oplus_{i<j}\mathbb{C}e_{i,j} \cong \mathbb{C}M_{n-2,2}$.

7. From all these isomorphisms of S_n-representations, deduce a decomposition of $S^2(V^{st})$ into irreducibles.

   Solution. On the one hand, we have
   $$V^p \otimes V^p \cong V^{st} \otimes V^{st} \oplus 2V^{st} \oplus V^{tr} \cong S^2(V^{st}) \oplus \Lambda^2(V^{st}) \oplus 2V^{st} \oplus V^{tr}.$$
   On the other hand, we have
   $$V^p \otimes V^p \cong S^2(V^p) \oplus \Lambda^2(V^p) \cong \mathbb{C}M_{n-2,2} \oplus V^p \oplus \Lambda^2(V^{st}) \oplus V^{st}$$
   $$\cong (V_{n-2,2} \oplus V^{st} \oplus V^{tr}) \oplus (V^{st} \oplus V^{tr}) \oplus \Lambda^2(V^{st}) \oplus V^{st} \cong V_{n-2,2} \oplus \Lambda^2(V^{st}) \oplus 3V^{st} \oplus 2V^{tr}.$$
   Since each representation of a finite group uniquely decomposes into irreducibles, one concludes
   $$S^2(V^{st}) \cong V_{n-2,2} \oplus V^{st} \oplus V^{tr}.$$

8. Using the interpretation of $M_{n-2,1,1}$ in terms of Young tabloids, prove
   $$V^p \otimes V^p \cong \mathbb{C}M_{n-2,1,1} \oplus V^p.$$

   Solution. A basis of $V^p \otimes V^p$ is given by $\{e_i \otimes e_j \,|\, 1 \le i,j \le n\}$. Similarly to the previous questions, one shows that the vectors $\{e_i \otimes e_i \,|\, 1 \le i \le n\}$ span a sub-representation isomorphic to V^p, and the vectors $\{e_i \otimes e_j \,|\, 1 \le i \ne j \le n\}$ span a sub-representation isomorphic to $\mathbb{C}M_{n-2,1,1}$. To establish the second isomorphism, send $e_i \otimes e_j$ to the Young tabloid of shape $(n-2,1,1)$, with the entry i in the second row, j in the third row, and all the remaining entries from $\{1,2,\dots,n\}$ in the first row.

9. Combine this with the isomorphisms of S_n-representations established above to show that the irrep $V_{n-2,1,1}$ is isomorphic to a direct summand of $\Lambda^2(V^{st})$.

   Solution. In class, we have seen that $V_{n-2,1,1}$ is isomorphic to a direct summand of $\mathbb{C}M_{n-2,1,1}$ (Theorem 10), and hence of $V^p \otimes V^p$ (Q8). Further, in the proof of Q7, we established the decomposition
   $$V^p \otimes V^p \cong V_{n-2,2} \oplus \Lambda^2(V^{st}) \oplus 3V^{st} \oplus 2V^{tr}.$$
   Here all the direct summands, except possibly $\Lambda^2(V^{st})$, are irreducible and non-isomorphic to $V_{n-2,1,1}$, since different Specht representations are non-isomorphic. So $V_{n-2,1,1}$ is isomorphic to a direct summand of $\Lambda^2(V^{st})$.

10. Determine the degree of the irrep $V_{n-2,1,1}$ using two methods:
    (a) first, by counting standard Young tableaux;
    (b) then, by the Hook length formula.

    Solution. In a standard Young tableau of shape $(n-2,1,1)$, the entry 1 has to lie in the top left cell. The only restrictions on the remaining cells are $c_{2,1} < c_{3,1}$ and $c_{1,2} < c_{1,3} < \ldots < c_{1,n-2}$. Here $c_{i,j}$ denotes the entry in the cell (i,j) of our Young tableau. Thus the standard Young tableaux of shape $(n-2,1,1)$ are in bijection with the choices of 2 elements from the set $\{2, 3, \ldots, n\}$, the elements to be placed below 1. This yields

$$\dim_{\mathbb{C}} V_{n-2,1,1} = \#\,\mathrm{SYT}_{n-2,1,1} = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}.$$

Alternatively, the Hook length formula gives

$$\dim_{\mathbb{C}} V_{n-2,1,1} = \frac{n!}{(n-3)!2!n} = \frac{(n-1)(n-2)}{2}.$$

$$\begin{array}{|c|c|c|c|c|}\hline n & n-3 & n-4 & \cdots & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

11. Conclusion: Prove

$$V_{n-2,1,1} \cong \Lambda^2(V^{st}).$$

Solution. Since $\dim_{\mathbb{C}} V^{st} = n-1$, one has

$$\dim_{\mathbb{C}} \Lambda^2(V^{st}) = \frac{(n-1)(n-2)}{2} = \dim_{\mathbb{C}} V_{n-2,1,1}.$$

Then the direct summand of $\Lambda^2(V^{st})$ isomorphic to $V_{n-2,1,1}$ has to be the whole $\Lambda^2(V^{st})$.

Exercise 2. We will next prove the S_n-representation isomorphism $V_{n-2,1,1} \cong \Lambda^2(V^{st})$, where $n \geq 3$, using an alternative method. For a permutation $\sigma \in S_n$, denote by $f(\sigma)$ the number of its fixed points.

1. Express the character of $\Lambda^2(V^{st})$ in terms of the function f.

   Solution. From the S_n-representation isomorphism $V^p \cong V^{st} \oplus V^{tr}$, and from $\chi^{V^p}(\sigma) = f(\sigma)$, follows

   $$\chi^{V^{st}}(\sigma) = \chi^{V^p}(\sigma) - \chi^{V^{tr}}(\sigma) = f(\sigma) - 1,$$

   hence

   $$\begin{aligned}\chi^{\Lambda^2(V^{st})}(\sigma) &= \frac{1}{2}(\chi^{V^{st}}(\sigma)^2 - \chi^{V^{st}}(\sigma^2)) = \frac{1}{2}((f(\sigma)-1)^2 - (f(\sigma^2)-1)) \\ &= \frac{1}{2}(f(\sigma)^2 - f(\sigma^2)) - f(\sigma) + 1.\end{aligned}$$

2. Given a permutation $\sigma \in S_n$, express the number of 1-cycles and the number of 2-cycles in its decomposition into disjoint cycles in terms of the function f.

   Solution. Given $\sigma \in S_n$, denote by $c_k(\sigma)$ the number of k-cycles in its decomposition into disjoint cycles. A fixed point of $\sigma \in S_n$ corresponds precisely to such a 1-cycle, giving

   $$c_1(\sigma) = f(\sigma).$$

   Further, in σ^2, a $(2k-1)$-cycle of σ squares to a $2k-1$-cycle of σ^2, and a $2k$-cycle of σ splits into two k-cycles. Thus $c_1(\sigma^2) = c_1(\sigma) + 2c_2(\sigma)$, and

   $$c_2(\sigma) = \frac{1}{2}(c_1(\sigma^2) - c_1(\sigma)) = \frac{1}{2}(f(\sigma^2) - f(\sigma)).$$

3. Use this and the Frobenius formula to express the character of $V_{n-2,1,1}$ in terms of the function f. Conclude.

   Solution. By the Frobenius formula, $\chi^{V_{n-2,1,1}}(\sigma)$ is the coefficient of the monomial $x_1^n x_2^2 x_3$ in the polynomial $\Delta(x_1,x_2,x_3)P_{\lambda'}(x_1,x_2,x_3)$, where
$$\Delta(x_1,x_2,x_3) = (x_1-x_2)(x_1-x_3)(x_2-x_3),$$
$$P_{\lambda'}(x_1,x_2,x_3) = (x_1^{\lambda'_1}+x_2^{\lambda'_1}+x_3^{\lambda'_1})\cdots(x_1^{\lambda'_{k'}}+x_2^{\lambda'_{k'}}+x_3^{\lambda'_{k'}}),$$
   and the permutation σ is of cycle type $\lambda' = (\lambda'_1,\dots,\lambda'_{k'})$. All monomials in $\Delta(x_1,x_2,x_3)$ $P_{\lambda'}(x_1,x_2,x_3)$ are products of a monomial in Δ and a monomial in $P_{\lambda'}$. Let us look at the factor $x_2^2x_3$ of our monomial $x_1^nx_2^2x_3$. For this factor, there are several possibilities with respect to this Δ–$P_{\lambda'}$ decomposition:
   (a) $x_2^2x_3$ comes entirely from Δ, where it enters with the coefficient 1. Then x_1^n should come from $P_{\lambda'}$, where it enters with the coefficient 1. Overall coefficient: $1\cdot 1 = 1$.
   (b) x_2x_3 comes from Δ, and x_2 from $P_{\lambda'}$. The coefficient of $x_1x_2x_3$ in Δ is $-1+1=0$, so we may ignore this case.
   (c) x_2^2 comes from Δ, and x_3 from $P_{\lambda'}$. The coefficient of $x_1x_2^2$ in Δ is -1. The coefficient of $x_1^{n-1}x_3$ in $P_{\lambda'}$ is the number of i with $\lambda'_i = 1$ (one can get x_3 only from the factors $(x_1+x_2+x_3)$; other factors give larger powers of x_3). But this is precisely $c_1(\sigma) = f(\sigma)$. Overall coefficient: $(-1)\cdot f(\sigma) = -f(\sigma)$.
   (d) x_2 comes from Δ, and x_2x_3 from $P_{\lambda'}$. The coefficient of $x_1^2x_2$ in Δ is 1. Similarly to the previous case, the coefficient of $x_1^{n-2}x_2x_3$ in $P_{\lambda'}$ is the number of $i \neq j$ with $\lambda'_i = \lambda'_j = 1$, which is $c_1(\sigma)(c_1(\sigma)-1) = f(\sigma)(f(\sigma)-1)$. Overall coefficient: $f(\sigma)^2 - f(\sigma)$.
   (e) x_3 comes from Δ, and x_2^2 from $P_{\lambda'}$. The coefficient of $x_1^2x_3$ in Δ is -1. The coefficient of $x_1^{n-2}x_2^2$ in $P_{\lambda'}$ is the number of $i<j$ with $\lambda'_i = \lambda'_j = 1$, plus the number of i with $\lambda'_i = 2$. These two choices correspond to the two occurrences of x_2 coming respectively from different and from the same factor of $P_{\lambda'}$. Overall coefficient: $(-1)\cdot(\frac{1}{2}c_1(\sigma)(c_1(\sigma)-1)+c_2(\sigma)) = -\frac{1}{2}(f(\sigma)(f(\sigma)-1)+(f(\sigma^2)-f(\sigma))) = -\frac{1}{2}(f(\sigma)^2 + f(\sigma^2)) + f(\sigma)$.
   (f) $x_2^2x_3$ comes entirely from $P_{\lambda'}$. This is impossible, since Δ does not contain the monomial x_1^3.

   So, the overall coefficient of $x_1^nx_2^2x_3$ in $\Delta P_{\lambda'}$ is
$$1 - f(\sigma) + (f(\sigma)^2 - f(\sigma)) + (-\frac{1}{2}(f(\sigma)^2 + f(\sigma^2)) + f(\sigma))$$
$$= \frac{1}{2}(f(\sigma)^2 - f(\sigma^2)) - f(\sigma) + 1 = \chi^{\Lambda^2(V^{st})}(\sigma).$$
   Since the representations $V_{n-2,1,1}$ and $\Lambda^2(V^{st})$ have the same characters, they are isomorphic.

Exercise 3.

1. Build the character table of the alternating group A_5 using that of the symmetric group S_5 (constructed in Lecture 12).

   Solution. As was explained in class, all irreps of A_5 are obtained from those of S_5 by restriction, followed by splitting in two the restrictions of the irreps V_λ with self-conjugate $\lambda = \lambda^t$. So the character table of the alternating group A_5 is obtained from that of the symmetric group S_5 by removing columns corresponding to odd permutations, leaving

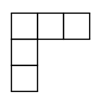
only one row from each couple of rows (V, $V' = V \otimes V^{\mathbf{sgn}}$), and, for each self-conjugate λ, splitting in two the row V_λ and the column corresponding to the cycle type $H(\lambda)$.
Recall the character table of S_5:

$\#\mathcal{C}$	1	10	20	30	24	15	20
S_5	Id	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
V^{tr}	1	1	1	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1	1	1	-1
V^{st}	4	2	1	0	-1	0	-1
$(V^{st})'$	4	-2	1	0	-1	0	1
$\Lambda^2(V^{st})$	6	0	0	0	1	-2	0
U	5	1	-1	-1	0	1	1
U'	5	-1	-1	1	0	1	-1

After all removals, one gets

$\#\mathcal{C}$	1	20	24	15
	Id	(123)	(12345)	(12)(34)
V^{tr}	1	1	1	1
V^{st}	4	1	-1	0
$\Lambda^2(V^{st})$	6	0	1	-2
U	5	-1	0	1

The only self-conjugate λ for $n = 5$ is $\lambda = (3, 1, 1)$, with $H(\lambda) = (5)$.



Indeed, since $n = 5$ is odd, a self-conjugate λ should have an odd number of cells on the main diagonal, and a tableau with ≥ 3 diagonal cells has $\geq 3^2 = 9$ cells in total.
So we should split the row corresponding to $V_{3,1,1} \cong \Lambda^2(V^{st})$ into two,

$$\mathrm{Res}_{A_5}^{S_5} \Lambda^2(V^{st}) \cong Z_1 \oplus Z_1'.$$

We should also split the column corresponding to the 5-cycle into two. Restricted to A_5, the conjugacy class of (12345) splits into two classes: that of (12345) and that of $(12)(12345)(12)^{-1} = (21345)$.

$\#\mathcal{C}$	1	20	12	12	15
A_5	Id	(123)	(12345)	(21345)	(12)(34)
$\mathrm{Res}\, V^{tr}$	1	1	1	1	1
$\mathrm{Res}\, V^{st}$	4	1	-1	-1	0
Z_1	3	0			-1
Z_1'	3	0			-1
$\mathrm{Res}\, U$	5	-1	0	0	1

The empty cells are determined by a formula seen in class. Here $m = \frac{1}{2}(n-k) = \frac{1}{2}(5-1) = 2$, where $k = 1$ is the number of parts in the partition $H(\lambda) = (5)$. Further,

$$\theta_\pm := \frac{1}{2}((-1)^m \pm \sqrt{(-1)^m \mu_1 \cdots \mu_k}) = \frac{1}{2}(1 \pm \sqrt{5}).$$

Our character table can now be completed as follows:

$\#\mathcal{C}$	1	20	12	12	15
A_5	Id	(123)	(12345)	(21345)	(12)(34)
V^{tr}	1	1	1	1	1
$\operatorname{Res} V^{st}$	4	1	-1	-1	0
Z_1	3	0	θ_+	θ_-	-1
Z_1'	3	0	θ_-	θ_+	-1
$\operatorname{Res} U$	5	-1	0	0	1

2. Decompose into irreducibles the symmetric and the alternating squares $S^2(\operatorname{Res}_{A_5}^{S_5} V^{st})$ and $\Lambda^2(\operatorname{Res}_{A_5}^{S_5} V^{st})$.

   Solution. We first compute the character of these reps, using formulas from Lecture 12:

$\mathcal{C}^2$	Id	(123)	(21345)	(12345)	Id
$\mathcal{C}$	Id	(123)	(12345)	(21345)	(12)(34)
$\operatorname{Res} V^{st}$	4	1	-1	-1	0
$\Lambda^2(\operatorname{Res} V^{st})$	6	0	1	1	-2
$S^2(\operatorname{Res} V^{st})$	10	1	0	0	2

   The only delicate point here is the computation of the class of $(12345)^2$ (the case of (21345) being similar). One has

$$(12345)^2 = (13524) = (245)(21345)(245)^{-1},$$

   which is conjugate to (21345) in A_5 since (245) is an even permutation, $(245) \in A_5$. Comparing with the previous table, or computing inner products of characters, one gets the desired decompositions:

$$\Lambda^2(\operatorname{Res} V^{st}) \cong Z_1 \oplus Z_1', \qquad S^2(\operatorname{Res} V^{st}) \cong \operatorname{Res} U \oplus \operatorname{Res} V^{st} \oplus V^{tr}.$$

3. Decompose into irreducibles the induced representations $\operatorname{Ind}_{A_4}^{A_5} V^{tr}$ and $\operatorname{Ind}_{A_4}^{A_5} \operatorname{Res}_{A_4}^{S_4} V^{st}$. You may use the character table of the alternating group A_4 from Lecture 21.

   Solution. First, let us compute the restrictions $\operatorname{Res}_{A_4}^{A_5}$ for all irreps of A_5. For this, one should remove from the character table of A_5 the columns corresponding to conjugacy classes of A_5 having empty intersection with A_4. Such classes are [(12345)] and [(21345)]:

	Id	(123)	(12)(34)
$\operatorname{Res}_{A_4}^{A_5} V^{tr}$	1	1	1
$\operatorname{Res}_{A_4}^{A_5} \operatorname{Res}_{A_5}^{S_5} V^{st}$	4	1	0
$\operatorname{Res}_{A_4}^{A_5} Z_1$	3	0	-1
$\operatorname{Res}_{A_4}^{A_5} Z_1'$	3	0	-1
$\operatorname{Res}_{A_4}^{A_5} \operatorname{Res}_{A_5}^{S_5} U$	5	-1	1

   Comparing with the character table of A_4:

A_4	Id	(123)	(132)	(12)(34)
V^{tr}	1	1	1	1
$\operatorname{Res}_{A_4}^{S_4} V^{st}$	3	0	0	-1
W_1	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1
W_1'	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1

one decomposes all the restrictions into irreps:

$$\operatorname{Res}^{A_5}_{A_4} V^{tr} \cong V^{tr}, \qquad \operatorname{Res}^{A_5}_{A_4} \operatorname{Res}^{S_5}_{A_5} V^{st} \cong \operatorname{Res} V^{st} \oplus V^{tr},$$
$$\operatorname{Res}^{A_5}_{A_4} Z_1 \cong \operatorname{Res}^{A_5}_{A_4} Z_1' \cong \operatorname{Res} V^{st}, \qquad \operatorname{Res}^{A_5}_{A_4} \operatorname{Res}^{S_5}_{A_5} U \cong \operatorname{Res} V^{st} \oplus W_1 \oplus W_1'.$$

By the Frobenius reciprocity law, $V \in \operatorname{Irrep}(A_5)$ enters into the decomposition of $\operatorname{Ind}^{A_5}_{A_4} V^{tr}$ into irreps with the multiplicity of V^{tr} in $\operatorname{Res}^{A_5}_{A_4} V$. Hence

$$\operatorname{Ind}^{A_5}_{A_4} V^{tr} \cong V^{tr} \oplus \operatorname{Res}^{S_5}_{A_5} V^{st}.$$

Similarly,

$$\operatorname{Ind}^{A_5}_{A_4} \operatorname{Res}^{S_4}_{A_4} V^{st} \cong \operatorname{Res}^{S_5}_{A_5} V^{st} \oplus Z_1 \oplus Z_1' \oplus \operatorname{Res}^{S_5}_{A_5} U.$$