Homework 3: Representations of S_n and A_n.

Instructions. Try to give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those.

Exercise 1. Our aim is to recover the irrep $V_{n-2,1,1}$ of the symmetric group S_n as the alternating square $\Lambda^2(V^{st})$, and the irrep $V_{n-2,2}$ as a direct summand of the symmetric square $S^2(V^{st})$.
First, take positive integers $p \geq q$. Consider the two-part partition (p,q) of $p+q$.

1. Prove that the following S_{p+q}-representations are isomorphic:
$$\mathbb{C}M_{p,q} \cong V_{p,q} \oplus \mathbb{C}M_{p+1,q-1},$$
using the Frobenius formula for the character of V_λ and its analogue for $\mathbb{C}M_\lambda$.
2. Deduce from this the degree of $V_{p,q}$. Compare with the result predicted by the Hook length formula.
3. Decompose the S_{p+q}-representation $\mathbb{C}M_{p,q}$ into irreducibles.

Now, take an integer $n \geq 4$.

4. As a particular case of the previous question, show that the following S_n-representations are isomorphic:
$$\mathbb{C}M_{n-2,2} \cong V_{n-2,2} \oplus V^{st} \oplus V^{tr}.$$
5. Recall the permutation representation $V^p = \oplus_{i=1}^n \mathbb{C}e_i$ of S_n, with $\sigma \cdot e_i = e_{\sigma(i)}$. Verify that the following S_n-representations are isomorphic:
$$V^p \otimes V^p \cong V^{st} \otimes V^{st} \oplus 2V^{st} \oplus V^{tr},$$
$$\Lambda^2(V^p) \cong \Lambda^2(V^{st}) \oplus V^{st}.$$
6. Using the interpretation of $M_{n-2,2}$ in terms of Young tabloids, prove
$$S^2(V^p) \cong \mathbb{C}M_{n-2,2} \oplus V^p,$$
and decompose the symmetric square $S^2(V^p)$ into irreducibles.
7. From all these isomorphisms of S_n-representations, deduce a decomposition of $S^2(V^{st})$ into irreducibles.
8. Using the interpretation of $M_{n-2,1,1}$ in terms of Young tabloids, prove
$$V^p \otimes V^p \cong \mathbb{C}M_{n-2,1,1} \oplus V^p.$$
9. Combine this with the isomorphisms of S_n-representations established above to show that the irrep $V_{n-2,1,1}$ is isomorphic to a direct summand of $\Lambda^2(V^{st})$.
10. Determine the degree of the irrep $V_{n-2,1,1}$ using two methods:
    (a) first, by counting standard Young tableaux;
    (b) then, by the Hook length formula.
11. Conclusion: Prove
$$V_{n-2,1,1} \cong \Lambda^2(V^{st}).$$

Exercise 2. We will next prove the S_n-representation isomorphism $V_{n-2,1,1} \cong \Lambda^2(V^{st})$, where $n \geq 3$, using an alternative method. For a permutation $\sigma \in S_n$, denote by $f(\sigma)$ the number of its fixed points.

1. Express the character of $\Lambda^2(V^{st})$ in terms of the function f.
2. Given a permutation $\sigma \in S_n$, express the number of 1-cycles and the number of 2-cycles in its decomposition into disjoint cycles in terms of the function f.
3. Use this and the Frobenius formula to express the character of $V_{n-2,1,1}$ in terms of the function f. Conclude.

Exercise 3.

1. Build the character table of the alternating group A_5 using that of the symmetric group S_5 (constructed in Lecture 12).
2. Decompose into irreducibles the symmetric and the alternating squares $S^2(\mathrm{Res}_{A_5}^{S_5} V^{st})$ and $\Lambda^2(\mathrm{Res}_{A_5}^{S_5} V^{st})$.
3. Decompose into irreducibles the induced representations $\mathrm{Ind}_{A_4}^{A_5} V^{tr}$ and $\mathrm{Ind}_{A_4}^{A_5} \mathrm{Res}_{A_4}^{S_4} V^{st}$. You may use the character table of the alternating group A_4 from Lecture 21.