Homework/Tutorial 2

Please hand in your work at the end of the tutorial. Make sure you put your name and student ID number on what you hand in. Please write your work in an intelligible way!

A complete solution to every question is worth 2 marks.

What this homework is about

You'll practise in finding the natural domain and the range of functions.

Reminder

The **natural domain** of a function f defined by a formula consists of all values of x for which $f(x)$ has a well defined real value. To find it, you might use the following algorithm:

1. Whenever you see an expression of type $\sqrt{g(x)}$ (or $\sqrt[2n]{g(x)}$ with $n \in \mathbb{N}$), determine all x for which $g(x) \geq 0$.
2. Whenever you see an expression of type $\frac{h(x)}{g(x)}$, determine all x for which $g(x) \neq 0$. Beware of nested fractions and of implicit fractions, like $\tan(x) = \frac{\sin(x)}{\cos(x)}$.
3. Determine the intersection of all the parts of $\mathbb{R}$ obtained in the previous points.

The **range** of a function f consists of all values $f(x)$ it assumes when x ranges over its domain.

Domains and ranges are best described as unions of non-intersecting intervals and rays, e.g., $(-\infty, -4) \cup (-4, -1] \cup [0, 3]$.

Given two functions f and g, their **composition** $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)),$$

so x must be in the domain of g and $g(x)$ in the domain of f for this to make sense.

Questions

1. Consider the functions $f(x) = \frac{x^2+2x}{x}$ and $g(x) = \sqrt{(x+1)^2}+1$. Show that these functions are different. Find a ray on which they are equal.

   Solution. These functions are different al least because they have different natural domains: $x = 0$ is in the domain of g, but not of f (division by zero). Alternatively, you can compare their values at any $x < -1$.
   Since $x^2 + 2x = x(x + 2)$, the function $f(x)$ simplifies as $x + 2$, $x \neq 0$. As for g, we get

   $$g(x) = |x + 1| + 1 = \begin{cases} x + 2, & x \geq -1, \\ -x, & x < -1. \end{cases}$$

   So, the functions coincide on $(0, +\infty)$. (Any sub-ray of $(0, +\infty)$, with a correct justification, is considered a correct answer.)

2. Consider the functions $f(x) = 2\sqrt{x-1} + \frac{1}{x-1}$ and $g(x) = -\frac{1}{x-1}$. Write the following functions in as simple a form as possible, and determine their domain:
   (a) $f + g$;
   (b) $\frac{g}{f}$.

Solution.

(a) $(f+g)(x) = f(x) + g(x) = 2\sqrt{x-1} + \frac{1}{x-1} - \frac{1}{x-1} = 2\sqrt{x-1}$. Its domain is the intersection of the domains of f and g. The domain of g is $[1, +\infty)$, since the square root makes sense only for non-negative numbers. The domain of g is $(1, +\infty)$: one gets $x \geq 1$ from the square root, and $x \neq 1$ from the denominator $x - 1$. The intersection of the two is $(1, +\infty)$. So, the final answer is

$$(f+g)(x) = 2\sqrt{x-1}, \qquad x > 1.$$

(b) $(d\frac{g}{f})(x) = \dfrac{g(x)}{f(x)} = \dfrac{-\frac{1}{x-1}}{2\sqrt{x-1} + \dfrac{1}{x-1}} = -\frac{1}{2\sqrt{x-1}(x-1)+1}$ whenever $x \neq 1$. The domain is the part of the intersection of the domains of f and g (here $(1, +\infty)$, as seen above) on which $f(x) \neq 0$. But f is defined only for $x > 1$, thus $f(x) = 2\sqrt{x-1} + \frac{1}{x-1} > 0 + 0 = 0$, and $f(x) \neq 0$ on the domain of f. The final answer is

$$(\frac{g}{f})(x) = -\frac{1}{2\sqrt{x-1}(x-1)+1}, \qquad x > 1.$$

3. Find the range of the function $f(x) = \frac{2x^2+1}{x^2+1}$.

Solution. We are looking for all real a for which the equation $\frac{2x^2+1}{x^2+1} = a$ has a solution in x. Since $x^2 + 1 > 0$ for all x (-0.5 points if you forget this verification!), we can multiply both sides by $x^2 + 1$, and get

$$2x^2 + 1 = (x^2 + 1)a,$$
$$x^2(2 - a) = a - 1.$$

If $a = 2$, this becomes $0 = 1$, which has no solution in x. Otherwise $x^2 = \frac{a-1}{2-a}$, which can be solved if and only if $\frac{a-1}{2-a} \geq 0$, which, by inspection of the sign of $a - 1$ and $2 - a$, means $a \in [1, 2)$. So, the range of our f is $[1, 2)$.

A faster solution, shown in class, is to write

$$\frac{2x^2+1}{x^2+1} = \frac{2(x^2+1)-1}{x^2+1} = \frac{2(x^2+1)}{x^2+1} - \frac{1}{x^2+1} = 2 - \frac{1}{x^2+1}.$$

The range of $x^2 + 1$ is $[1, +\infty)$, so the range of $\frac{1}{x^2+1}$ is $(0, 1]$, and the range of $2 - \frac{1}{x^2+1}$ is $[1, 2)$.

4. Consider the functions $f(x) = \sqrt{x} + x + 1$ and $g(x) = x + \frac{1}{x}$. Determine their compositions $f \circ g$ and $g \circ f$, and their domains.

Solution. $(f \circ g)(x) = f(g(x)) = \sqrt{x + \frac{1}{x}} + x + \frac{1}{x} + 1$. The domain of f is $[0, +\infty)$ because of the square root, and the domain of g is $(-\infty, 0) \cup (0, +\infty)$ because of $\frac{1}{x}$. Thus, we are looking for all $x \neq 0$ (domain of g) for which $x + \frac{1}{x} \geq 0$ (domain of f). Now, $x + \frac{1}{x} = \frac{x^2+1}{x}$. Since $x^2 + 1 > 0$ for all x, we get $x + \frac{1}{x} \geq 0$ iff $x > 0$. Summarizing, we conclude that the domain of $f \circ g$ is $(0, +\infty)$.

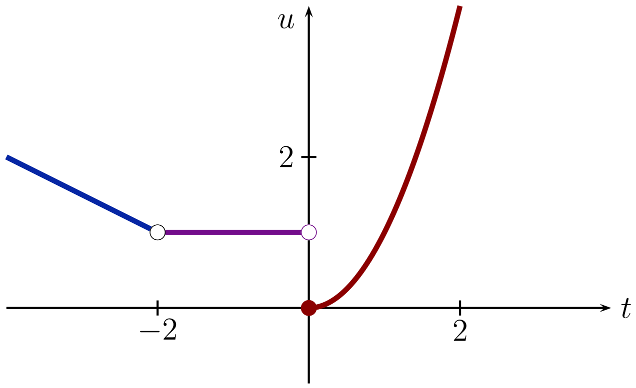
Further, $(g \circ f)(x) = g(f(x)) = g(x) = \sqrt{x} + x + 1 + \frac{1}{\sqrt{x}+x+1}$. We are looking for all $x \geq 0$ (domain of f) for which $\sqrt{x} + x + 1 \neq 0$ (domain of g). Now, for all $x \geq 0$ one has $\sqrt{x} + x + 1 \geq 0 + 0 + 1 > 0$. Conclusion: that the domain of $g \circ f$ is $[0, +\infty)$.

5. Consider the piecewise defined function

$$u(t) = \begin{cases} t^2, & t \geq 0, \\ 1, & -2 < t < 0, \\ -\frac{1}{2}t, & t < -2. \end{cases}$$

What are its domain and its range? Plot its graph. Also, plot the graph of the function $v(t) = u(t+1) - 2$.

Solution. By definition, the domain of u is $(-\infty, -2) \cup (-2, 0) \cup [0, +\infty) = (-\infty, -2) \cup (-2, +\infty)$. For $t < -2$, is takes all values in $(1, +\infty)$. For $-2 < t < 0$, it takes only one value, 1. For $t \geq 0$, is takes all values in $[0, +\infty)$. Summarizing, the range of u is $[0, +\infty)$. The graph of u consist of 3 distinct pieces:



The graph of $v(t) = u(t+1) - 2$ is obtained from that of u by shifting by 1 to the left and by 2 to the bottom: