Homework 2: Character tables

Instructions. Try to give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those.

Remark. We propose two conceptually different approaches to the study of the two groups below. For the first one, we describe some (and as it turns out, all) of its irreps, and use them to deduce important properties of the group (such as the complete list of its element). For the second one, we first explore the group itself, and then build the study of its irreps on our observations.

Exercise 1. Character table for the quaternion group Q

Consider the group Q defined by generators and relations as follows:

$$Q = \langle \overline{\mathbf{1}}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \,|\, \overline{\mathbf{1}}^2 = \mathbf{1}, \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = \overline{\mathbf{1}} \rangle.$$

Here $\mathbf{1}$ is the neutral element of Q. We will also use notation $\overline{x} = \overline{\mathbf{1}}x$ for all $x \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

Remark: If you do not recognise these relations, I strongly recommend you to read about quaternions, and include Brougham Bridge into your next Dublin walk!

1. Determine all representations of Q of degree 1.

   Solution. We are looking for all maps $\rho\colon \{\overline{\mathbf{1}}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \to \mathbb{C}^*$ satisfying

   $$\rho(\overline{\mathbf{1}})^2 = 1, \qquad \rho(\boldsymbol{i})^2 = \rho(\boldsymbol{j})^2 = \rho(\boldsymbol{k})^2 = \rho(\boldsymbol{i})\rho(\boldsymbol{j})\rho(\boldsymbol{k}) = \rho(\overline{\mathbf{1}}). \tag{1}$$

   This implies

   $$\rho(\overline{\mathbf{1}}) = \rho(\overline{\mathbf{1}})^3 = \rho(\boldsymbol{i})^2\rho(\boldsymbol{j})^2\rho(\boldsymbol{k})^2 = (\rho(\boldsymbol{i})\rho(\boldsymbol{j})\rho(\boldsymbol{k}))^2 = \rho(\overline{\mathbf{1}})^2.$$

   Since $\rho(\overline{\mathbf{1}}) \neq 0$, this yields $\rho(\overline{\mathbf{1}}) = 1$. Then (1) rewrites as

   $$\rho(\boldsymbol{i}) = \pm 1,\ \rho(\boldsymbol{j}) = \pm 1,\ \rho(\boldsymbol{k}) = \pm 1,\ \rho(\boldsymbol{i})\rho(\boldsymbol{j})\rho(\boldsymbol{k}) = 1.$$

   This system has 4 solutions: either $\rho(\boldsymbol{i}) = \rho(\boldsymbol{j}) = \rho(\boldsymbol{k}) = 1$ (the trivial rep.), or one of them is 1 and the two remaining ones are -1.

2. Check that the formulas

   $$\rho(\overline{\mathbf{1}}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(\boldsymbol{i}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \rho(\boldsymbol{j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(\boldsymbol{k}) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   define a representation (V, ρ) of Q of degree 2.

   Solution. All these matrices are invertible. A direct computation using matrix multiplication shows that they satisfy relations (1), where 1 in $\rho(\overline{\mathbf{1}})^2 = 1$ is replaced with the unit matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

3. Is this representation irreducible?

   Solution. Yes. Indeed, its non-trivial sub-rep. U must be of degree 1: $U = \mathbb{C}u$ for some $u \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. This u has to be an eigenvector for the four matrices from Q2. The matrix $\rho(\boldsymbol{i})$ has distinct eigenvalues $i, -i$, with eigenvectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ respectively. Thus u is a non-zero multiple of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. But none of these is an eigenvector for $\rho(\boldsymbol{j})$.

4. Is Q an abelian group?

   Solution. No: all irreps of a finite abelian group are of degree 1.

5. Consider the subset $X = \{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \overline{\mathbf{1}}, \overline{\boldsymbol{i}}, \overline{\boldsymbol{j}}, \overline{\boldsymbol{k}}\}$ of Q. Using the previous points, prove that that these 8 elements are pairwise distinct.

   Solution. It suffices to show that for any pair of elements from X, there is an irrep of Q taking different values on them. Degree 1 irreps take the same value on $\overline{x}$ and x for all $x \in \{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Apart from these pairs, they distinguish all the elements of X. Now, for the representation (V, ρ) from Q2,

$$\rho(\overline{x}) = \rho(\overline{\mathbf{1}}x) = \rho(\overline{\mathbf{1}})\rho(x) = -\rho(x) \neq \rho(x)$$

   for all $x \in \{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

6. Show that $\overline{\mathbf{1}}$ lies in the center of Q (i.e., commutes with all elements of Q).

   Solution. $\overline{\mathbf{1}}$ clearly commutes with itself. Further,

$$\overline{\mathbf{1}}\boldsymbol{i} = \overline{\mathbf{1}}\,\overline{\mathbf{1}}^2 = \overline{\mathbf{1}}^3 = \overline{\mathbf{1}}^2\,\overline{\mathbf{1}} = \boldsymbol{i}\overline{\mathbf{1}}.$$

   The same argument works for $\boldsymbol{j}$ and $\boldsymbol{k}$. Since $\overline{\mathbf{1}}$ commutes with the generators of Q, it has to commute with all elements of Q.

7. Use this to simplify the products $x\overline{x}$ and $\overline{x}x$ for all $x \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

   Solution. $x\overline{x} = x\overline{\mathbf{1}}x = \overline{\mathbf{1}}xx = \overline{\mathbf{1}}\,\overline{\mathbf{1}} = \mathbf{1}, \qquad \overline{x}x = \overline{\mathbf{1}}xx = \overline{\mathbf{1}}\,\overline{\mathbf{1}} = \mathbf{1}.$

8. Verify the relations $\boldsymbol{ij} = \boldsymbol{k}$ and $\boldsymbol{ji} = \overline{\boldsymbol{k}}$.

   Solution. $\boldsymbol{ij} = \boldsymbol{ij}\mathbf{1} = \boldsymbol{ijk}\overline{\boldsymbol{k}} = \overline{\mathbf{1}}\,\overline{\boldsymbol{k}} = \overline{\mathbf{1}}\,\overline{\mathbf{1}}\boldsymbol{k} = \boldsymbol{k}$. Further, $\boldsymbol{ijji} = \boldsymbol{i}\overline{\mathbf{1}}\boldsymbol{i} = \overline{\mathbf{1}}\boldsymbol{ii} = \overline{\mathbf{1}}\,\overline{\mathbf{1}} = \mathbf{1}$, so $\boldsymbol{ji} = (\boldsymbol{ij})^{-1} = (\boldsymbol{k})^{-1} = \overline{\boldsymbol{k}}$.

9. Using the previous points, prove that the product of any two elements from X lies in X. Summarise your computations in a multiplication table for X.

   Solution. For any $x \in \{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ and $y \in X$, one has $\overline{\mathbf{1}}\,\overline{x} = \overline{\mathbf{1}}\,\overline{\mathbf{1}}x = x$, $\overline{x}y = \overline{\mathbf{1}}xy$, and $y\overline{x} = y\overline{\mathbf{1}}x = \overline{\mathbf{1}}yx$. Further, $\mathbf{1}$ is the neutral element of Q. Thus it suffices to know how to multiply elements from the smaller set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Recall the defining relations for Q: $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \overline{\mathbf{1}}$. Moreover, the relation $\boldsymbol{ij} = \boldsymbol{k}$ from Q8 implies $\boldsymbol{kij} = \boldsymbol{kk} = \overline{\mathbf{1}} = \boldsymbol{jj}$, hence $\boldsymbol{ki} = \boldsymbol{j}$. Repeating the same argument, one gets $\boldsymbol{jk} = \boldsymbol{i}$. Taking inverses on each side, one concludes $\boldsymbol{ik} = \overline{\boldsymbol{j}}$ and $\boldsymbol{kj} = \overline{\boldsymbol{i}}$.

	$\boldsymbol{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\overline{\boldsymbol{1}}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{k}}$
$\boldsymbol{1}$	$\boldsymbol{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\overline{\boldsymbol{1}}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{k}}$
$\boldsymbol{i}$	$\boldsymbol{i}$	$\overline{\boldsymbol{1}}$	$\boldsymbol{k}$	$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{i}}$	$\boldsymbol{1}$	$\overline{\boldsymbol{k}}$	$\boldsymbol{j}$
$\boldsymbol{j}$	$\boldsymbol{j}$	$\overline{\boldsymbol{k}}$	$\overline{\boldsymbol{1}}$	$\boldsymbol{i}$	$\overline{\boldsymbol{j}}$	$\boldsymbol{k}$	$\boldsymbol{1}$	$\overline{\boldsymbol{i}}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{j}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{1}}$	$\overline{\boldsymbol{k}}$	$\overline{\boldsymbol{j}}$	$\boldsymbol{i}$	$\boldsymbol{1}$
$\overline{\boldsymbol{1}}$	$\overline{\boldsymbol{1}}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{k}}$	$\boldsymbol{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{i}}$	$\boldsymbol{1}$	$\overline{\boldsymbol{k}}$	$\boldsymbol{j}$	$\boldsymbol{i}$	$\overline{\boldsymbol{1}}$	$\boldsymbol{k}$	$\overline{\boldsymbol{j}}$
$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{j}}$	$\boldsymbol{k}$	$\boldsymbol{1}$	$\overline{\boldsymbol{i}}$	$\boldsymbol{j}$	$\overline{\boldsymbol{k}}$	$\overline{\boldsymbol{1}}$	$\boldsymbol{i}$
$\overline{\boldsymbol{k}}$	$\overline{\boldsymbol{k}}$	$\overline{\boldsymbol{j}}$	$\boldsymbol{i}$	$\boldsymbol{1}$	$\boldsymbol{k}$	$\boldsymbol{j}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{1}}$

10. Show that the inverse of any element from X lies in X.

    Solution. The inverses can be determined from the above table:

x	$\boldsymbol{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\overline{\boldsymbol{1}}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{k}}$
x^{-1}	$\boldsymbol{1}$	$\overline{\boldsymbol{i}}$	$\overline{\boldsymbol{j}}$	$\overline{\boldsymbol{k}}$	$\overline{\boldsymbol{1}}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$

11. Deduce that X is actually the whole set Q. Thus in Q9 you constructed a multiplication table for the whole group Q.

    Solution. The non-empty set X is stable under taking products and inverses. It is thus a sub-group of Q. Moreover, it contains all the generators $\overline{\boldsymbol{1}}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of Q. Thus it has to be the whole group Q.

12. Verify that $\overline{x}$ is a conjugate of x for all $x \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

    Solution. $\boldsymbol{i}^{-1}\boldsymbol{j}\boldsymbol{i} = \overline{\boldsymbol{i}}\,\overline{\boldsymbol{k}} = \boldsymbol{i}\boldsymbol{k} = \overline{\boldsymbol{j}}$. Similarly, $\boldsymbol{j}^{-1}\boldsymbol{k}\boldsymbol{j} = \overline{\boldsymbol{k}}$ and $\boldsymbol{k}^{-1}\boldsymbol{i}\boldsymbol{k} = \overline{\boldsymbol{i}}$.

13. Describe all conjugacy classes of Q.

    Solution. By Q11 and Q12, there are at most $8-3=5$ conjugacy classes in Q. On the other hand, we have described 5 pairwise non-isomorphic irreps of Q, so $\#\operatorname{Conj}(Q) = \#\operatorname{Irrep}(Q) \geq 5$. Combining these two observations, one gets $\#\operatorname{Conj}(Q) = 5$. Thus there are no conjugacy relations in Q other than those from Q12. Conclusion:

$$\operatorname{Conj}(Q) = \{[\boldsymbol{1}], [\overline{\boldsymbol{1}}], [\boldsymbol{i}], [\boldsymbol{j}], [\boldsymbol{k}]\}.$$

    The first two classes are of size 1, and the remaining three of size 2: $[x] = \{x, \overline{x}\}$.

14. Construct a character table of Q.

    Solution. $\#\operatorname{Irrep}(Q) = \#\operatorname{Conj}(Q) = 5$, so Q1 and Q2 describe all irreps of Q by explicit matrices. It remains to compute the traces of these matrices:

$\#\mathcal{C}$	1	1	2	2	2
V \ $\mathcal{C}$	$[\boldsymbol{1}]$	$[\overline{\boldsymbol{1}}]$	$[\boldsymbol{i}]$	$[\boldsymbol{j}]$	$[\boldsymbol{k}]$
$V_0 = V^{tr}$	1	1	1	1	1
V_1	1	1	-1	-1	1
V_2	1	1	1	-1	-1
V_3	1	1	-1	1	-1
V	2	-2	0	0	0
V^{reg}	8	0	0	0	0

Double-checking: $\chi^{V^{reg}} = \sum_{i=0}^{3} \chi^{V_i} + 2\chi^{V}$, as expected.

15. For the representation V from Q2, decompose into irreps its tensor square $V \otimes V$, its symmetric square $S^2(V)$, and its alternating square $\Lambda^2(V)$.

Solution. Using the formulas

$$\chi^{V\otimes V} = \chi^V \chi^V,\ \chi^{S^2(V)}(g) = \frac{\chi^V(g)^2 + \chi^V(g^2)}{2},\ \chi^{\Lambda^2(V)}(g) = \frac{\chi^V(g)^2 - \chi^V(g^2)}{2},$$

one computes

$\mathcal{C}^2$	$[\mathbf{1}]$	$[\mathbf{1}]$	$[\overline{\mathbf{1}}]$	$[\overline{\mathbf{1}}]$	$[\overline{\mathbf{1}}]$
$\#\mathcal{C}$	1	1	2	2	2
$V \backslash \mathcal{C}$	$[\mathbf{1}]$	$[\overline{\mathbf{1}}]$	$[\boldsymbol{i}]$	$[\boldsymbol{j}]$	$[\boldsymbol{k}]$
V	2	-2	0	0	0
$V \otimes V$	4	4	0	0	0
$S^2(V)$	3	3	-1	-1	-1
$\Lambda^2(V)$	1	1	1	1	1

This immediately yields

$$\Lambda^2(V) \cong V^{tr}.$$

Further, $(\chi^{V\otimes V}, \chi^{V\otimes V}) = \frac{1}{8}(1\cdot 4^2 + 1\cdot 4^2 + 2\cdot 0^2 + 2\cdot 0^2 + 2\cdot 0^2) = \frac{1}{8}32 = 4$, which has two decompositions into a sum of squares: $4 = 2^2$ and $4 = 1+1+1+1$. Thus the degree 4 representation $V\otimes V$ decomposes as a direct sum of either 2 isomorphic irreps of degree 2, or 4 pairwise non-isomorphic irreps of degree 1. The first case is impossible, since $V \otimes V$ has as a direct summand $\Lambda^2(V) \cong V^{tr}$. The second case yields

$$V \otimes V \cong \oplus_{i=0}^{3} V_i.$$

Finally, using $V \otimes V \cong \Lambda^2(V) \oplus S^2(V)$, one concludes

$$S^2(V) \cong \oplus_{i=1}^{3} V_i.$$

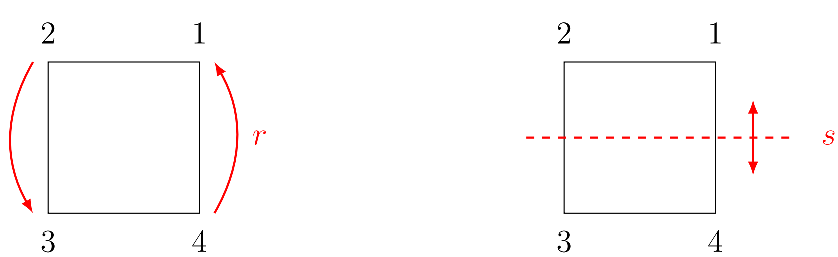
Double-checking: $\chi^{S^2(V)} = \sum_{i=1}^{3} \chi^{V_i}$, as expected.

16. What are the degrees of the three representations from the previous point?

Solution. $4, 3, 1$. More generally, in class we have seen that for a degree n representation, $\deg(V\otimes V) = n^2$, $\deg(S^2(V)) = \frac{n(n+1)}{2}$, $\deg(\Lambda^2(V)) = \frac{n(n-1)}{2}$.

Exercise 2. Character table for the dihedral group D_8

Let D_8 be the group of symmetries of a square S. Denote by r and by s respectively a $\frac{\pi}{2}$-rotation and a reflection, as shown in the figure:



1. How many elements are there in D_8? Write them all explicitly in terms of the generators r and s.

   Solution. A symmetry of S is uniquely determined by where it sends one vertex (say, vertex 1), and whether or not it preserves the orientation of S. Moreover, all the $4 \cdot 2 = 8$ combinations of these parameters are realisable, by the following symmetries:

   $$\mathrm{Id}, r, r^2, r^3, s, sr^2, sr, sr^3.$$

   Thus this is a complete list of the elements of D_8, and $\#D_8 = 8$.
   These symmetries can also be described geometrically. The first four are all the possible rotations. The next two are the horizontal and the vertical symmetries. And the last two are symmetries w.r.t. the two diagonals. The first four symmetries preserve the orientation, and the last four do not.

2. Find an abelian subgroup of D_8 of index 2.

   Solution. Since $r^4 = \mathrm{Id}$, the elements Id, r, r^2, r^3 form a subgroup of D_8. More precisely, it is a cyclic subgroup of size 4, so it is abelian and has index $\frac{8}{4} = 2$ in D_8.

3. Is D_8 abelian?

   Solution. No: $srs^{-1} = srs = r^3$, since srs sends vertex 1 to 4 and preserves the orientation of S (since r preserves the orientation and s does not).

4. What do the previous points tell you about the irreps of D_8?

   Solution. First, D_8 has an abelian subgroup of index 2, thus (by a theorem seen in class) the degree of any of its irreps is ≤ 2. Second, D_8 itself is not abelian, so (by another theorem seen in class) it has at least one irrep of degree 2.

5. Determine the number and the degrees of the irreps of D_8.

   Solution. Denote by d_i the degrees of the k irreps V_i of D_8. As usual, we choose $V_0 = V^{tr}$. One has $\sum_{i=0}^{k-1} d_i^2 = \#D_8 = 8$; $1 \leq d_i \leq 2$ and $\exists d_i = 2$ by the previous question; and $d_0 = 1$. These restrictions leave only one possibility (up to a reordering of the irreps): $k = 5$, $d_0 = d_1 = d_2 = d_3 = 1$, $d_4 = 2$.

6. Check the following relations in D_8: $r^4 = s^2 = 1$, $srs = r^{-1}$.

   Solution. Track where all these symmetries send vertex 1, and whether or not they preserve the orientation of S.

   Remark: In fact, by a cardinality comparison, one obtains a presentation of D_8 by generators and relations:
   $$D_8 \cong \langle r, s \,|\, r^4 = s^2 = 1,\ srs = r^{-1} \rangle.$$

7. Show that for any degree 1 representation (V_i, ρ_i) of D_8, $\rho_i(s) \in \{\pm 1\}$ and $\rho_i(r) \in \{\pm 1\}$.

   Solution. By the previous question, all degree 1 irreps satisfy $\rho_i(r)^4 = \rho_i(s)^2 = 1$, and $\rho_i(s)\rho_i(r)\rho_i(s) = \rho_i(r)^{-1}$. The first relation yields $\rho_i(s) \in \{\pm 1\}$. The second one yields $\rho_i(r)^2 = \rho_i(s)^{-2} = 1$, hence $\rho_i(r) \in \{\pm 1\}$.

8. Knowing the total number of degree 1 representations, explain why in the previous point all the 4 possibilities have to be realised for some (V_i, ρ_i).

   Solution. Since r and s generate D_8 (by Q1), the values of ρ_i on these two elements uniquely determine its values on the whole group D_8. Since D_8 has 4 distinct degree 1 irreps, all the 4 possible combinations of their values on r and s have to be realised.

9. Describe all conjugacy classes of D_8.

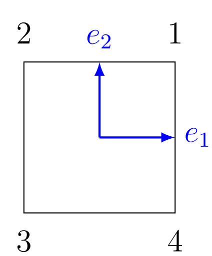
   Solution. $\#\,\mathrm{Conj}(D_8) = \#\,\mathrm{Irrep}(D_8) = 5$. Using relations from Q6, one computes $srs^{-1} = srs = r^{-1} = r^3$, $rsr^{-1} = s^{-1}r^{-1}r^{-1} = sr^2$, and $r(sr)r^{-1} = rs = s^{-1}r^{-1} = sr^3$. Thus $r \sim r^3, s \sim sr^2, sr \sim sr^3$, yielding al most 5 conjugacy classes. No new conjugacy relations are possible, since they would further decrease the number of conjugacy classes. Conclusion:
   $$\mathrm{Conj}(D_8) = \{[\mathrm{Id}], [r^2], [r], [s], [sr]\}.$$
   The first two classes are of size 1, and the remaining three of size 2: $[r] = \{r, r^3\}$, $[s] = \{s, sr^2\}$, $[sr] = \{sr, sr^3\}$.

10. Construct a character table of D_8.

    Solution. The characters of all degree 1 irreps V_i can be deduced from Q7-Q8. The character of the remaining degree 2 irrep is then deduced from $V^{reg} \cong (\oplus_{i=0}^{3} V_i) \oplus 2V$.

$\#\mathcal{C}$	1	1	2	2	2
V \ $\mathcal{C}$	$[\mathrm{Id}]$	$[r^2]$	$[r]$	$[s]$	$[sr]$
$V_0 = V^{tr}$	1	1	1	1	1
V_1	1	1	-1	-1	1
V_2	1	1	1	-1	-1
V_3	1	1	-1	1	-1
V	2	-2	0	0	0
V^{reg}	8	0	0	0	0

11. The symmetries of our square S extend to linear transformations of the whole plane $\mathbb{R}^2$ containing it, by fixing the center of S at the origin of $\mathbb{R}^2$. This yields a degree 2 representation U of D_8. Write its matrices in the following basis:



    Solution. It suffices to describe the matrices M_r and M_s for the generators r, s of D_8; all other elements of D_8 are products of these generators and their inverses, and, since a representation is a group homomorphism, the matrices for all these elements can be expressed in terms of M_r and M_s.

Now, the rotation r sends e_1 to e_2, and e_2 to $-e_1$. The horizontal symmetry s sends e_1 to itself, and e_2 to $-e_2$. Their matrices are then

$$M_r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark. In fact, this yields a real representation of D_8, $\rho\colon D_8 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$. But since $\mathbb{R}$ injects into $\mathbb{C}$, it can also be regarded as a complex representation.

12. Use them to compute the character of U, and then to decompose U into irreps.

    Solution. $\chi^U(\mathrm{Id}) = \dim_{\mathbb{C}}(U) = 2$, $\chi^U(r^2) = \mathrm{tr}(M_r^2) = \mathrm{tr}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -2$, $\chi^U(r) = \mathrm{tr}(M_r) = 0$, $\chi^U(s) = \mathrm{tr}(M_s) = 0$, $\chi^U(sr) = \mathrm{tr}(M_s M_r) = \mathrm{tr}\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right) = 0$. Since $\chi^U = \chi^V$, one concludes $U \cong V$.

13. Considering the action of the symmetries of S on its vertices, explain how to interpret D_8 as a subgroup of the symmetric group S_4. Denote the inclusion map by $\iota\colon D_8 \to S_4$.

    Solution. Enumerate the vertices of S, for example as in the picture at the beginning of the exercise. Each symmetry from D_8 permutes these 4 vertices, and to a composition of symmetries corresponds a composition of permutations. So one obtains a group homomorphism $\iota\colon D_8 \to S_4$. It is injective, since the new position of all the vertices uniquely determines the symmetry. Thus D_8 is isomorphic to its image $\iota(D_8) \subset S_4$, which is a subgroup of S_4.

14. For all $g \in D_8$, write down explicitly the corresponding permutation $\iota(g)$.

    Solution. Using the same vertex numbering as above, one has $\iota(\mathrm{Id}) = \mathrm{Id}$, $\iota(r) = (1234)$, $\iota(r^2) = (13)(24)$, $\iota(r^3) = (1432)$, $\iota(s) = (14)(23)$, $\iota(sr) = (24)$, $\iota(sr^2) = (12)(34)$, $\iota(sr^3) = (13)$.

15. Using characters, for all irreps V of S_4 (listed in Tutorial 2), decompose into irreps the corresponding representation $\iota^*(V)$ of D_8. (See Tutorial 1 for the construction of ι^*.)

    Solution. For $(V, \rho) \in \mathrm{Irrep}(S_4)$ and $g \in Q_8$, $\chi^{\iota^*(V)}(g) = \mathrm{tr}(\rho(\iota(g)) = \chi^V(\iota(g))$. Using this, one computes the characters of all $\iota^*(V)$ (irreducible characters are repeated here for convenience):

$\iota\mathcal{C}$	Id	(12)(34)	(1234)	(12)(34)	(12)
$\mathcal{C}$ / V	[Id]	$[r^2]$	$[r]$	$[s]$	$[sr]$
$\iota^* V^{tr}$	1	1	1	1	1
$\iota^* V^{\mathbf{sgn}}$	1	1	-1	1	-1
$\iota^* V^{st}$	3	-1	-1	-1	1
$\iota^* V^{st,\mathbf{sgn}}$	3	-1	1	-1	-1
$\iota^* W$	2	2	0	2	0
$V_0 = V^{tr}$	1	1	1	1	1
V_1	1	1	-1	-1	1
V_2	1	1	1	-1	-1
V_3	1	1	-1	1	-1
V	2	-2	0	0	0

Comparing the two tables, or using the inner product of characters, one concludes:

$$\iota^* V^{tr} \cong V^{tr}, \quad \iota^* V^{\mathbf{sgn}} \cong V_3, \quad \iota^* W \cong V^{tr} \oplus V_3, \quad \iota^* V^{st} \cong V \oplus V_1, \quad \iota^* V^{st,\mathbf{sgn}} \cong V \oplus V_2.$$

16. Are the groups Q and D_8 isomorphic? (*Hint:* You can for example compare the number of square roots of the neutral element in both groups.) How similar are their character tables? Conclude.

Solution. These groups are not isomorphic, since in Q the neutral element $\mathbf{1}$ has only 2 square roots, $\mathbf{1}$ and $\overline{\mathbf{1}}$ (all other elements square to $\overline{\mathbf{1}}$), and in D_8 the neutral element Id has 4 square roots, Id, r^2, and sr^α for $0 \leq \alpha \leq 3$. However, these two groups have the same character tables (up to a permutation of rows and columns). Conclusion: The character table does not determine the group uniquely.

Remark. Alternatively, one can argue that D_8 has a degree 2 irrep with real matrices in a good basis (Q11), whereas for the degree 2 irrep (V, ρ) of Q, the linear automorphism $\rho(\boldsymbol{i})$ of V has eigenvalues $\pm i \notin \mathbb{R}$.