Homework 2: Character tables

Instructions. Try to give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those.

Exercise 1. Character table for the quaternion group Q

Consider the group Q defined by generators and relations as follows:

$$Q = \langle \overline{\mathbf{1}}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \,|\, \overline{\mathbf{1}}^2 = \mathbf{1}, \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = \overline{\mathbf{1}} \rangle.$$

Here $\mathbf{1}$ is the neutral element of Q. We will also use notation $\overline{x} = \overline{\mathbf{1}}x$ for all $x \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

Remark: If you do not recognise these relations, I strongly recommend you to read about quaternions, and include Brougham Bridge into your next Dublin walk!

1. Determine all representations of Q of degree 1.
2. Check that the formulas

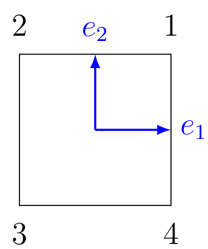
$$\rho(\overline{\mathbf{1}}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(\boldsymbol{i}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \rho(\boldsymbol{j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(\boldsymbol{k}) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   define a representation (V, ρ) of Q of degree 2.
3. Is this representation irreducible?
4. Is Q an abelian group?
5. Consider the subset $X = \{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, \overline{\mathbf{1}}, \overline{\boldsymbol{i}}, \overline{\boldsymbol{j}}, \overline{\boldsymbol{k}}\}$ of Q. Using the previous points, prove that that these 8 elements are pairwise distinct.
6. Show that $\overline{\mathbf{1}}$ lies in the center of Q (i.e., commutes with all elements of Q).
7. Use this to simplify the products $x\overline{x}$ and $\overline{x}x$ for all $x \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.
8. Verify the relations $\boldsymbol{ij} = \boldsymbol{k}$ and $\boldsymbol{ji} = \overline{\boldsymbol{k}}$.
9. Using the previous points, prove that the product of any two elements from X lies in X. Summarise your computations in a multiplication table for X.
10. Show that the inverse of any element from X lies in X.
11. Deduce that X is actually the whole set Q. Thus in Q9 you constructed a multiplication table for the whole group Q.
12. Verify that $\overline{x}$ is a conjugate of x for all $x \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.
13. Describe all conjugacy classes of Q.
14. Construct a character table of Q.
15. For the representation V from Q2, decompose into irreps its tensor square $V \otimes V$, its symmetric square $S^2(V)$, and its alternating square $\Lambda^2(V)$.
16. What are the degrees of the three representations from the previous point?

Exercise 2. Character table for the dihedral group D_8

Let D_8 be the group of symmetries of a square S. Denote by r and by s respectively a $\frac{\pi}{2}$-rotation and a reflection, as shown in the figure:

1. How many elements are there in D_8? Write them all explicitly in terms of the generators r and s.
2. Find an abelian subgroup of D_8 of index 2.
3. Is D_8 abelian?
4. What do the previous points tell you about the irreps of D_8?
5. Determine the number and the degrees of the irreps of D_8.
6. Check the following relations in D_8: $r^4 = s^2 = 1$, $srs = r^{-1}$.
7. Show that for any degree 1 representation (V_i, ρ_i) of D_8, $\rho_i(s) \in \{\pm 1\}$ and $\rho_i(r) \in \{\pm 1\}$.
8. Knowing the total number of degree 1 representations, explain why in the previous point all the 4 possibilities have to be realised for some (V_i, ρ_i).
9. Describe all conjugacy classes of D_8.
10. Construct a character table of D_8.
11. The symmetries of our square S extend to linear transformations of the whole plane $\mathbb{R}^2$ containing it, by fixing the center of S at the origin of $\mathbb{R}^2$. This yields a degree 2 representation U of D_8. Write its matrices in the following basis:



12. Use them to compute the character of U, and then to decompose U into irreps.
13. Considering the action of the symmetries of S on its vertices, explain how to interpret D_8 as a subgroup of the symmetric group S_4. Denote the inclusion map by $\iota\colon D_8 \to S_4$.
14. For all $g \in D_8$, write down explicitly the corresponding permutation $\iota(g)$.
15. Using characters, for all irreps V of S_4 (listed in Tutorial 2), decompose into irreps the corresponding representation $\iota^*(V)$ of D_8. (See Tutorial 1 for the construction of ι^*.)
16. Are the groups Q and D_8 isomorphic? (*Hint:* You can for example compare the number of square roots of the neutral element in both groups.) How similar are their character tables? Conclude.