Homework 1:
Representations and characters of finite groups

Instructions. Try to give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those. The marks for the first 3 exercises sum up to 100; Exercise 4 gives you a bonus.

Exercise 1. Consider the group G of invertible upper triangular matrices $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ over $\mathbb{C}$. Let $\rho\colon G \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ be the inclusion map.

1. Show that $(\mathbb{C}^2, \rho)$ is a representation of G.
2. What is the degree of ρ?
3. Prove that ρ has precisely one sub-representation of degree 1.
4. Is this sub-representation isomorphic to a representation we have already seen?
5. Is ρ irreducible?
6. Is ρ indecomposable?
7. Under what condition is an indecomposable representation of a group necessarily irreducible? Does this condition hold for our group G?

Remark: This example shows that for having the complete reducibility for representations, it is essential to work with finite groups.

Solution.

1. By definition, $\forall M \in G$, the matrix $\rho(M) = M$ is invertible, so ρ takes values in the group $\mathrm{Mat}^*_{2\times 2}(\mathbb{C})$ of invertible matrices. Further, $\forall M_1, M_2 \in G$, $\rho(M_1 M_2) = M_1 M_2 = \rho(M_1)\rho(M_2)$, so ρ is a group morphism $G \to \mathrm{Mat}^*_{2\times 2}(\mathbb{C})$. It thus endows $\mathbb{C}^2$ with a G-representation structure.
2. The degree of ρ is $\dim_{\mathbb{C}}(\mathbb{C}^2) = 2$.
3. The 1-dimensional sub-space $\mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is a sub-representation: $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = a\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Suppose that ρ has another sub-representation $V = \mathbb{C}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, with $y \neq 0$ (otherwise it coincides with the first one). Then $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) - \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in V$, i.e., $\left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right) \in V$. But then V contains two linearly independent vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right)$, which contradicts $\dim_{\mathbb{C}}(V) = 1$.
4. No. The only general representation of degree 1 we have seen is the trivial one. It is isomorphic only to itself, since for any isomorphism $\phi\colon W \to U$, $\phi\,\mathrm{Id}_W\,\phi^{-1} = \mathrm{Id}_U$. And in our case some matrices act non-trivially: $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = 2\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.
5. No. It has a non-zero sub-representation $\mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ which is proper, since $\dim_{\mathbb{C}}(\mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = 1 < 2 = \dim_{\mathbb{C}}(\mathbb{C}^2)$.
6. Yes. Its non-trivial decomposition should have the form $\mathbb{C}^2 = V_1 \oplus V_2$, where the V_i are sub-representations of degree 1. But we have seen that $\mathbb{C}^2$ has only one such sub-representation.
7. When the group is finite. (It is a consequence of Maschke's theorem.) Our group G is infinite, which explains why it can have a reducible but indecomposable representation.

Exercise 2. Consider the permutation representation $V_n^{perm} = \oplus_{i=1}^n \mathbb{C}e_i$ of the symmetric group S_n, with S_n acting by $\sigma \cdot e_i = e_{\sigma(i)}$. Suppose $n \geq 3$.

1. Prove that V_n^{perm} has precisely one sub-representation of degree 1. Denote it by L.
2. Identify the representation L.
3. Define a linear map $\varepsilon\colon V_n^{perm} \to \mathbb{C}$ by $\varepsilon(e_i) = 1$ for all i. Show that $V_n^{st} = \operatorname{Ker}\varepsilon$ is a sub-representation of V_n^{perm}. It is called the *standard representation* of S_n.
4. What is the degree of V_n^{st}?
5. Give a basis of V_n^{st}.
6. Verify that V_n^{st} is irreducible.
7. Show that the S_n-representations V_n^{perm} and $V^{tr} \oplus V_n^{st}$ are isomorphic. (Here V^{tr} is the trivial degree 1 representation.)
8. Express the characters of V_n^{perm} and V_n^{st} in terms of fixed point sets for permutations.
9. A representation $\rho\colon G \to \operatorname{Aut}_{\mathbb{C}}(V)$ is called *faithful* if the map ρ is injective. Which of our representations V^{tr}, V_n^{perm}, V_n^{st} are faithful?

Solution.

1. Suppose that $\mathbb{C}e$ is a sub-representation of degree 1, with $e = \sum_{i=1}^n \alpha_i e_i$, $\alpha_i \in \mathbb{C}$, and not all the α_i are zero. Suppose that $\alpha_i \neq \alpha_j$ for some i, j. Then the sub-representation $\mathbb{C}e$ contains $e - (ij)\cdot e = (\alpha_i - \alpha_j)(e_i - e_j)$, and thus $e_i - e_j$. Since $n \geq 3$, there is a $k \notin \{i, j\}$, $1 \leq k \leq n$. Then $\mathbb{C}e$ also contains $(ik)\cdot(e_i - e_j) = e_k - e_j$ which is not proportional to $e_i - e_j$, and thus $\mathbb{C}e$ cannot have dimension 1. Conclusion: $\alpha_i = \alpha_j$ for all i, j, that is, $\mathbb{C}e = \mathbb{C}(\sum_{i=1}^n e_i)$. This is indeed a sub-representation: $\forall \sigma \in S_n$, $\sigma \cdot (\sum_{i=1}^n e_i) = \sum_{i=1}^n e_{\sigma(i)} = \sum_{i=1}^n e_i$.
2. It is the trivial representation: above we showed that all $\sigma \in S_n$ act on it trivially.
3. Since $\sigma \cdot (\sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \alpha_i e_{\sigma(i)}$,
$$\varepsilon(\sum_{i=1}^n \alpha_i e_i) = \sum_{i=1}^n \alpha_i = \varepsilon(\sigma \cdot (\sum_{i=1}^n \alpha_i e_i)). \tag{1}$$
Hence $\varepsilon(\sum_{i=1}^n \alpha_i e_i) = 0$ implies $\varepsilon(\sigma \cdot (\sum_{i=1}^n \alpha_i e_i)) = 0$, and $\operatorname{Ker}\varepsilon$ is S_n-invariant.
Alternatively, one can use (1) to show that $\varepsilon \in \operatorname{Hom}_{S_n}(V_n^{perm}, V^{tr})$, and recall that a kernel of a morphism of representations is itself a representation (seen in class).
4. $\dim_{\mathbb{C}}(\operatorname{Ker}\varepsilon) = \dim_{\mathbb{C}}(V_n^{perm}) - \dim_{\mathbb{C}}(\operatorname{Im}\varepsilon) = n - 1$. We used that $\operatorname{Im}\varepsilon = \mathbb{C}$, since it is a $\mathbb{C}$-linear sub-space of $\mathbb{C}$ containing $\varepsilon(e_1) = 1$.
5. $\operatorname{Ker}\varepsilon = \{\sum_{i=1}^n \alpha_i e_i | \sum_{i=1}^n \alpha_i = 0, \alpha_i \in \mathbb{C}\} = \{\sum_{i=1}^{n-1} \alpha_i e_i - (\sum_{i=1}^{n-1} \alpha_i) e_n | \alpha_i \in \mathbb{C}\} = \{\sum_{i=1}^{n-1} \alpha_i (e_i - e_n) | \alpha_i \in \mathbb{C}\}$. Thus the $n-1$ vectors $e_i - e_n$, $1 \leq i \leq n-1$, span $\operatorname{Ker}\varepsilon$. To show that they form a basis of $\operatorname{Ker}\varepsilon$, we need to check that they are also linearly independent: $\alpha_i(e_i - e_n) = 0$ implies, by looking at the coefficient of each e_i, that all α_i vanish.
6. Suppose that V_n^{st} has a non-zero sub-representation U. Take a non-zero element $e = \sum_{i=1}^n \alpha_i e_i$ of U. One has $\alpha_i \neq \alpha_j$ for some i, j — otherwise $\varepsilon(e) = 0$ would imply $e = 0$. Repeating the argument from Question 1, one gets $e_i - e_j \in U$. Then, for each $1 \leq m \leq n-1$, take any permutation σ sending i to m and j to n. Then $e_m - e_n = \sigma \cdot (e_i - e_j) \in U$. So U contains a basis of V_n^{st}. Thus U has to be the whole V_n^{st}.
7. Since $L \cong V^{tr}$, it suffices to show $V_n^{perm} = L \oplus V_n^{st}$. First, $\dim_{\mathbb{C}}(V_n^{perm}) = n = \dim_{\mathbb{C}}(V_n^{st}) + \dim_{\mathbb{C}}(L)$. Second, $e \in L \cap V_n^{st}$ implies $e = \alpha(\sum_{i=1}^n e_i)$ and $0 = \varepsilon(e) = \alpha n$. But then $\alpha = 0$, hence $e = 0$. So $L \cap V_n^{st} = \{0\}$.
8. We have seen that $\chi^{V_n^{perm}}(\sigma) = \#\{1, 2, \ldots, n\}^\sigma$ (the number of elements in $\{1, 2, \ldots, n\}$ fixed by σ). Then $\chi^{V_n^{st}}(\sigma) = \chi^{V_n^{perm}}(\sigma) - \chi^{V_n^{tr}}(\sigma) = \#\{1, 2, \ldots, n\}^\sigma - 1$.
9. V^{tr} is clearly not faithful: it sends all $\sigma \in S_n$ to the same element $1 \in \mathbb{C}^*$. V^{perm} is faithful: $\rho^{perm}(\sigma) = \mathrm{Id}_{V^{perm}}$ means $\forall i, \sigma \cdot e_i = e_i$, that is $\sigma(i) = i$; but the only permutation fixing

all i is Id. V_n^{st} is also faithful: since $\rho^{perm} \cong \rho^{st} \oplus \rho^{tr}$, and $\rho^{tr}(\sigma) = 1$ for all σ, $\rho^{st}(\sigma) = \mathrm{Id}_{V^{st}}$ implies $\rho^{perm}(\sigma) = \mathrm{Id}_{V^{perm}}$, hence, as shown above, $\sigma = \mathrm{Id}$.

Exercise 3. In this exercise we will learn how to deform irreps for getting new ones.

Consider a finite group G and its representations (V, ρ) of degree k, and $(\mathbb{C}, \omega)$ of degree 1.

1. Verify that the map
$$G \to \mathrm{Aut}_{\mathbb{C}}(V),$$
$$g \mapsto \omega(g)\rho(g),$$
defines another G-representation on V. Denote it by (V^ω, ρ_ω).
2. How are the characters χ^ρ and χ^{ρ_ω} related?
3. When do these characters coincide?
4. Show that the G-representation V is irreducible if and only if V^ω is so.
5. When are these two representations isomorphic?

Example: Specialise to the group $G = S_n$, $n \geq 4$, with the standard representation (V_n^{st}, ρ) from Exercise 2. Put $\omega(\sigma) = \mathbf{sgn}(\sigma)$.

6. Verify that the signature $\mathbf{sgn}$ defines a degree 1 representation of S_n.
7. Show that $V_n^{st,\mathbf{sgn}}$ is an irreducible S_n-representation, not isomorphic to V_n^{st}.

Solution.

1. The map ω takes values in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$, which is the group $\mathbb{C}^*$ of non-zero complex numbers. Indeed, a $\mathbb{C}$-linear map from $\mathbb{C}$ to itself is necessarily a multiplication by a non-zero scalar. Then $\forall g \in G$, $\omega(g)\rho(g)$ is a $\mathbb{C}$-linear automorphism of V multiplied by a non-zero scalar, which is still in $\mathrm{Aut}_{\mathbb{C}}(V)$. Hence ρ_ω is a well-defined map $G \to \mathrm{Aut}_{\mathbb{C}}(V)$. It remains to check that this is a group morphism: $\rho_\omega(gh) = \omega(gh)\rho(gh) = (\omega(g)\omega(h))\rho(g)\rho(h) = (\omega(g)\rho(g))(\omega(h)\rho(h)) = \rho_\omega(g)\rho_\omega(h)$.
2. $\chi^{\rho_\omega}(g) = \mathrm{tr}(\rho_\omega(g)) = \mathrm{tr}(\omega(g)\rho(g)) = \omega(g)\,\mathrm{tr}(\rho(g)) = \omega(g)\chi^\rho(g)$. Thus, $\chi^{\rho_\omega} = \omega\chi^\rho$, where maps $G \to \mathbb{C}$ are multiplied pointwise.
3. When $\chi^\rho(g) = \omega(g)\chi^\rho(g)$ for all g. That is, $\forall g \in G$, either $\chi^\rho(g)$ should vanish, or $\omega(g)$ should be 1.
4. (V, ρ) is irreducible $\iff (\chi^\rho, \chi^\rho) = 1 \iff \frac{1}{\#G}\sum_{g\in G} |\chi^\rho(g)|^2 = 1 \iff \frac{1}{\#G}\sum_{g\in G} |\chi^{\rho_\omega}(g)|^2 = 1 \iff (\chi^{\rho_\omega}, \chi^{\rho_\omega}) = 1 \iff (V^\omega, \rho_\omega)$ is irreducible. Here we used $|\chi^{\rho_\omega}(g)| = |\omega(g)||\chi^\rho(g)| = |\chi^\rho(g)|$. Indeed, our group is finite, so $g^k = 1$ for some k, hence $\omega(g)^k = 1$, and $\omega(g)$ is a root of unity.
5. When their characters coincide.
6. The signature $\mathbf{sgn}$ takes values $\pm 1 \in \mathbb{C}^*$. Further, it is a multiplicative map: $\forall \sigma_1, \sigma_2 \in S_n$, $\mathbf{sgn}(\sigma_1\sigma_2) = \mathbf{sgn}(\sigma_1)\,\mathbf{sgn}(\sigma_2)$, since if permutations σ_i can be written using n_i transpositions, then $\sigma_1\sigma_2$ can be written using $n_1 + n_2$ of them.
7. In Exercise 2, we have seen that V_n^{st} is irreducible, and $\chi^{V_n^{st}}(\sigma) = \#\{1, 2, \ldots, n\}^\sigma - 1$. By Question 4, $V_n^{st,\mathbf{sgn}}$ is then irreducible as well. Now, consider the transposition $(12) \in S_n$. One has $\chi^{V_n^{st}}((12)) = \#\{1, 2, \ldots, n\}^{(12)} - 1 = (n-2) - 1 = n - 3 > 0$, and $\mathbf{sgn}((12)) = -1$. Then, by Questions 3 and 5, $V_n^{st,\mathbf{sgn}} \ncong V_n^{st}$.