Homework/Tutorial 10

Marks: each question is worth 2 marks.

What this homework is about

You will learn how to compute definite integrals, and explore their relation with indefinite integrals and area computation.

Reminder

The area $A$ under the graph of a continuous non-negative function $f$ on an interval $[a, b]$ is defined as follows. For all integers $N$,

1) Divide $[a, b]$ into $N$ subintervals $I_1, \ldots, I_N$ of equal size $\Delta_N = \frac{b-a}{N}$.
2) On each $I_k$, choose some point $x_k^*$.
3) On each $I_k$, approximate the figure under the graph of $f$ by the rectangle $\Delta_N \times f(x_k^*)$.
4) Compute the total area $\sum_{k=1}^{N} f(x_k^*) \Delta_N$ of these small rectangles.

The area $A$ is defined as the limiting value of these sums: $A = \lim_{N \to +\infty} \sum_{k=1}^{N} f(x_k^*) \Delta_N$.

The definite integral $\int_{a}^{b} f(x) \, dx$ of a function $f$ on $[a, b]$ is the limit of similar sums (called Riemann sums), where the subintervals $I_k$ are no longer of the same size, but become arbitrarily small: $\max |I_k| \to 0$. The function is called (Riemann) integrable on $[a, b]$ if this limit exists.

For a continuous non-negative $f$, $\int_{a}^{b} f(x) \, dx$ computes the area under the graph of $f$ on $[a, b]$.

The following formulas are useful for computing Riemann sums:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Definite integrals are usually evaluated using indefinite integrals, thanks to

The Fundamental Theorem of Calculus. Let $f$ be continuous on $[a, b]$. Then:

Part 1: $f$ is Riemann integrable on $[a, b]$, and $\int_{a}^{b} f(x) \, dx = \left[ \int f(x) \, dx \right]_{a}^{b} = F(b) - F(a)$, where $F$ is any antiderivative of $f$ on $[a, b]$;

Part 2: the function $F(x) = \int_{a}^{x} f(t) \, dt$ is an antiderivative of $f$ on $[a, b]$.

Properties of definite integrals:

- linearity: $\int_{a}^{b} (c_1 f_1(x) + \cdots + c_n f_n(x)) \, dx = c_1 \int_{a}^{b} f_1(x) \, dx + \cdots + c_n \int_{a}^{b} f_n(x) \, dx$;
- $u$-substitution: $\int_{a}^{b} f(g(x)) g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$;
- integration by parts: $\int_{a}^{b} f'(x) g(x) \, dx = \left[ f(x) g(x) \right]_{a}^{b} - \int_{a}^{b} [f(x) g'(x)] \, dx$;
- $\int_{a}^{a} f(x) \, dx = 0$; $\int_{b}^{a} f(x) \, dx = - \int_{a}^{b} f(x) \, dx$;
- additivity: $\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx$;
- monotony: $f(x) \geq g(x)$ on $[a, b] \implies \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx$. 
Questions

1. Consider the function \( F(x) = \int_{1}^{x} \frac{dt}{\sqrt{3t+1}}. \)
   
   (a) What is its natural domain?
   (b) Compute \( F'(x) \).
   (c) Explain why \( F \) increases on its domain.
   (d) Compute \( F(0) \).

Solution.

(a) The function \( f(t) = \frac{1}{\sqrt{3t+1}} \) is defined and continuous when \( 3t+1 > 0 \), that is, for \( t \in (-\frac{1}{3}, +\infty) \). Since a continuous function is always Riemann integrable, and \( 1 \in (-\frac{1}{3}, +\infty) \), the natural domain of \( F \) is also \((-\frac{1}{3}, +\infty)\).

(b) By the Fundamental Theorem of Calculus, \( F(x) = \int_{1}^{x} f(t) dt \) is an antiderivative of \( f \), so \( F'(x) = f(x) = \frac{1}{\sqrt{3x+1}} \).

(c) Since \( F'(x) = f(x) = \frac{1}{\sqrt{3x+1}} > 0 \) for all \( x \in (-\frac{1}{3}, +\infty) \), the function \( F \) increases on its domain.

A second explanation: \( F(x) \) is the area under the graph of the continuous non-negative function \( f \) on \([1, x]\), so it increases when the interval \([1, x]\) extends to the right.

(d) We'll use the substitution \( u = 3t + 1 \), with \( du = 3dt \), \( u(1) = 4 \), \( u(0) = 1 \):

\[
F(0) = \int_{1}^{0} \frac{dt}{\sqrt{3t+1}} = \int_{4}^{1} \frac{1}{3\sqrt{u}} du = \frac{1}{3} \left[ 2\sqrt{u} \right]_{4}^{1} = \frac{2}{3} (1 - 2) = -\frac{2}{3}.
\]

2. Compute the following definite integrals:

   (a) \( \int_{1}^{0} x^2 \sqrt{x^3 - 1} \, dx \);
   
   (b) \( \int_{1}^{e^2} x \ln x \, dx \).

Solution.

(a) We'll use the substitution \( u(x) = x^3 - 1 \), with \( du = 3x^2 \, dx \), \( u(1) = 0 \), \( u(0) = -1 \):

\[
\int_{1}^{0} x^2 \sqrt{x^3 - 1} \, dx = \int_{0}^{-1} \frac{du}{3} \left[ \frac{1}{2} \frac{u^{\frac{1}{2}}}{u} \right]_{0}^{-1} = \frac{5}{18} (1 + 1) = \frac{5}{18}.
\]

(b) We'll use integration by parts with \( f(x) = \frac{x^2}{2} \) and \( g(x) = \ln x \):

\[
\int_{1}^{e^2} x \ln x \, dx = \left[ \frac{x^2}{2} \ln x \right]_{1}^{e^2} - \int_{1}^{e^2} \frac{x^2}{2} \, dx = \left( \frac{e^4}{2} - \frac{1}{2} \right) - \int_{1}^{e^2} \frac{x^2}{2} \, dx = \frac{3}{4} e^4 + \frac{1}{4}.
\]

3. Compute

\[
\int_{1}^{\frac{\pi}{4}} f(x) \, dx, \quad \text{where} \quad f(x) = \frac{\arctan(x)}{\sqrt{x^2 + 1} \cos(x)},
\]

without finding the antiderivative of \( f \).

(Hint. Compare \( \int_{1}^{0} f(x) \, dx \) and \( \int_{0}^{\frac{\pi}{4}} f(x) \, dx \)).
4. Consider the functions \( f(x) = \sqrt{x^2 + 1} \) and \( \cos(x) \) are even, and \( \arctan(x) \) is odd, therefore \( f \) is odd. Keeping this in mind, let us make the following substitution in \( \int_{0}^{1} f(x) \, dx \):

\[
\text{Let } u(x) = -x, \quad du = -dx, \quad u(0) = 0, \quad u(1) = 1.
\]

We get

\[
\int_{0}^{1} f(x) \, dx = \int_{u(0)}^{u(1)} f(u) (-du) = \int_{0}^{1} (-f(u)) (-du) = \int_{0}^{1} f(u) \, du.
\]

So,

\[
\int_{1}^{0} f(x) \, dx = \int_{0}^{1} f(x) \, dx + \int_{1}^{0} f(x) \, dx = \int_{0}^{1} f(x) \, dx + \int_{0}^{1} f(x) \, dx = \int_{1}^{0} f(x) \, dx = 0.
\]

**Remark.** Using the same argument, one can show that for any odd function \( f \) Riemann integrable on \([-a, a]\), we have \( \int_{-a}^{0} f(x) \, dx = 0 \).

4. Consider the functions \( f(x) = \sin\left(\frac{x}{2}\right) \) and \( g(x) = x \) on \([0, 1]\).

   (a) Show that on \([0, 1]\), the graph of \( f \) lies above the graph of \( g \). 
      (Hint. Consider the difference \( h(x) = f(x) - g(x) \). To see that \( h(x) \geq 0 \) on \([0, 1]\), find the minimum of \( h \) on \([0, 1]\).)

   (b) Compute the area between the two graphs on \([0, 1]\).

**Solution.** Consider the difference of the two functions, \( h(x) = f(x) - g(x) = \sin\left(\frac{x}{2}\right) - x \). To determine its minimal value, we need to compute its derivative:

\[
h'(x) = \sin\left(\frac{x}{2}\right)' - x' = \cos\left(\frac{x}{2}\right) \cdot \frac{\pi}{2} - 1.
\]

When \( x \) changes between 0 and 1, \( \cos\left(\frac{x}{2}\right) \) decreases from \( \cos(0) = 1 \) to \( \cos\left(\frac{\pi}{2}\right) = 0 \), so \( h'(x) \) decreases from \( \frac{\pi}{2} - 1 \) (which is positive) to \(-1\). This means that \( h(x) \) increases up to a certain point (namely, \( x = \frac{\pi}{2} \arccos\left(\frac{x}{2}\right) \)) and then decreases. Therefore, it takes its minimal value at one of its endpoints, 0 or 1. A direct computation yields

\[
h(0) = \sin(0) - 0 = 0, \quad h(1) = \sin\left(\frac{\pi}{2}\right) - 1 = 0.
\]

Hence this minimal value is 0, and \( h(x) \geq 0 \) on \([0, 1]\). In other words, \( f(x) \geq g(x) \) on \([0, 1]\), so the graph of \( f \) lies above the graph of \( g \). As a result, the area \( A \) between the two graphs is the difference between the area under the graph of \( f \) and area under the graph of \( g \):

\[
A = \int_{0}^{1} f(x) \, dx - \int_{0}^{1} g(x) \, dx = \int_{0}^{1} \sin\left(\frac{x}{2}\right) \, dx - \int_{0}^{1} x \, dx = \left[ -\frac{2}{\pi} \cos\left(\frac{\pi}{2}\right) \right]_{0}^{1} \left[ -\frac{x^2}{2} \right]_{0}^{1}
\]

\[
= -\frac{2}{\pi} (\cos\left(\frac{\pi}{2}\right) - \cos(0)) - \left(\frac{1}{2} - 0\right) = \frac{2}{\pi} - \frac{1}{2}.
\]

5. Compute \( \int_{0}^{1} x^3 \, dx \) using two methods:

   (a) explicit Riemann sums, where you divide the interval \([0, 1]\) into \( N \) equal parts and in each part choose as the point \( x_k^* \) the left endpoint;

   (b) the relation with the indefinite integral.

Compare the answers obtained.

**Solution.**

(a) According to the algorithm recalled in the Reminder, take a natural number \( N \) and divide the interval \([0, 1]\) into \( N \) parts \([0, \frac{1}{N}], \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\) of equal size \( \frac{1}{N} \). On each
subinterval $[\frac{k-1}{N}, \frac{k}{N}]$, choose its left endpoint $\frac{k-1}{N}$. Then the corresponding Riemann sum for the function $x^3$ on $[0,1]$ becomes

$$0^3 \cdot \frac{1}{N} + \left(\frac{1}{N}\right)^3 \cdot \frac{1}{N} + \cdots + \left(\frac{N-1}{N}\right)^3 \cdot \frac{1}{N} = \frac{1}{N^4} \sum_{k=1}^{N-1} k^3 = \frac{1}{N^4} \frac{(N-1)^2 N^2}{4} = \frac{1}{4} (1 - \frac{1}{N})^2.$$ 

Here we applied the formula $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for $n = N - 1$. When $N$ gets arbitrary large, $\frac{1}{N}$ approaches 0, so the numbers $\frac{1}{4} (1 - \frac{1}{N})^2$ approach $\frac{1}{4}$. Thus $\int_0^1 x^3 \, dx = \frac{1}{4}$.

(b) $\int_0^1 x^3 \, dx = \left[ \frac{x^4}{4} \right]_0^1 = \frac{1}{4} (1^4 - 0^4) = \frac{1}{4}$.

The two methods for computing definite integrals yield the same result, but the second one demands much less efforts!