Homework/Tutorial 10

Marks: each question is worth 2 marks.

What this homework is about

You will learn how to compute definite integrals, and explore their relation with indefinite integrals and area computation.

Reminder

The **area** A under the graph of a continuous non-negative function f on an interval $[a,b]$ is defined as follows. For all integers N,

1) Divide $[a,b]$ into N subintervals $I_1, \dots I_N$ of equal size $\Delta_N = \frac{b-a}{N}$.
2) On each I_k, choose some point x_k^*.
3) On each I_k, approximate the figure under the graph of f by the rectangle $\Delta_N \times f(x_k^*)$.
4) Compute the total area $\sum_{k=1}^N f(x_k^*)\Delta_N$ of these small rectangles.

The area A is defined as the limiting value of these sums: $A = \lim\limits_{N\to+\infty} \sum\limits_{k=1}^{N} f(x_k^*)\Delta_N$.

The **definite integral** $\int_a^b f(x)\,dx$ of a function f on $[a,b]$ is the limit of similar sums (called **Riemann sums**), where the subintervals I_k are no longer of the same size, but become arbitrarily small: $\max\limits_k |I_k| \underset{N\to+\infty}{\longrightarrow} 0$. The function is called **(Riemann) integrable** on $[a,b]$ if this limit exists. For a continuous non-negative f, $\int_a^b f(x)\,dx$ computes the area under the graph of f on $[a,b]$.

The following formulas are useful for computing Riemann sums:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Definite integrals are usually evaluated using indefinite integrals, thanks to

The Fundamental Theorem of Calculus. Let f be continuous on $[a,b]$. Then:

Part 1: f is Riemann integrable on $[a,b]$, and $\displaystyle\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = F(b)-F(a)$, where F is any antiderivative of f on $[a,b]$;

Part 2: the function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f on $[a,b]$.

Properties of definite integrals:

- linearity: $\displaystyle\int_a^b (c_1 f_1(x)+\cdots+c_n f_n(x))\,dx = c_1\int_a^b f_1(x)\,dx+\cdots+c_n\int_a^b f_n(x)\,dx$;
- u-substitution: $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$;
- integration by parts: $\displaystyle\int_a^b f'(x)g(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b [f(x)g'(x)]\,dx$;
- $\displaystyle\int_a^a f(x)\,dx = 0; \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$;
- additivity: $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$;
- monotony: $f(x)\geq g(x)$ on $[a,b]$ $\implies$ $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Questions

1. Consider the function $F(x) = \int_1^x \frac{dt}{\sqrt{3t+1}}$.
   (a) What is its natural domain?
   (b) Compute $F'(x)$.
   (c) Explain why F increases on its domain.
   (d) Compute $F(0)$.

   Solution.
   (a) The function $f(t) = \frac{1}{\sqrt{3t+1}}$ is defined and continuous when $3t+1 > 0$, that is, for $t \in (-\frac{1}{3}, +\infty)$. Since a continuous function is always Riemann integrable, and $1 \in (-\frac{1}{3}, +\infty)$, the natural domain of F is also $(-\frac{1}{3}, +\infty)$.
   (b) By the Fundamental Theorem of Calculus, $F(x) = \int_1^x f(t)\,dt$ is an antiderivative of f, so $F'(x) = f(x) = \frac{1}{\sqrt{3x+1}}$.
   (c) Since $F'(x) = f(x) = \frac{1}{\sqrt{3x+1}} > 0$ for all $x \in (-\frac{1}{3}, +\infty)$, the function F increases on its domain.
   A second explanation: $F(x)$ is the area under the graph of the continuous non-negative function f on $[1, x]$, so it increases when the interval $[1, x]$ extends to the right.
   (d) We'll use the substitution $u = 3t + 1$, with $du = 3dt$, $u(1) = 4$, $u(0) = 1$:
   $$F(0) = \int_1^0 \frac{dt}{\sqrt{3t+1}} = \int_4^1 \frac{du}{3\sqrt{u}} = \frac{1}{3}\left[2\sqrt{u}\right]_4^1 = \frac{2}{3}(1-2) = -\frac{2}{3}.$$

2. Compute the following definite integrals:

   (a) $\int_1^0 x^2 \sqrt[5]{x^3-1}\,dx$;

   (b) $\int_1^{e^2} x \ln x\,dx$.

   Solution.
   (a) We'll use the substitution $u(x) = x^3 - 1$, with $du = 3x^2\,dx$, $u(1) = 0$, $u(0) = -1$:
   $$\int_1^0 x^2 \sqrt[5]{x^3-1}\,dx = \int_0^{-1} \sqrt[5]{u}\,\frac{du}{3} = \frac{1}{3}\left[\frac{5}{6}u^{\frac{6}{5}}\right]_0^{-1} = \frac{5}{18}(1-0) = \frac{5}{18}.$$
   (b) We'll use integration by parts with $f(x) = \frac{x^2}{2}$ and $g(x) = \ln x$:
   $$\int_1^{e^2} x \ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^{e^2} - \int_1^{e^2} \frac{x^2}{2}\frac{dx}{x} = \left(\frac{e^4}{2}\ln(e^2) - \frac{1^2}{2}\ln 1\right) - \int_1^{e^2} \frac{x}{2}\,dx$$
   $$= \left(\frac{e^4}{2}\cdot 2 - 0\right) - \left[\frac{1}{4}x^2\right]_1^{e^2} = e^4 - \frac{1}{4}(e^4 - 1) = \frac{3}{4}e^4 + \frac{1}{4}.$$

3. Compute
   $$\int_1^{-1} f(x)\,dx, \quad \text{where } f(x) = \frac{\arctan(x)}{\sqrt{x^2+1}\cos(x)},$$
   without finding the antiderivative of f.
   (*Hint.* Compare $\int_1^0 f(x)\,dx$ and $\int_0^{-1} f(x)\,dx$.)

Solution. The functions $\sqrt{x^2+1}$ and $\cos(x)$ are even, and $\arctan(x)$ is odd, therefore f is odd. Keeping this in mind, let us make the following substitution in $\int_0^{-1} f(x)\,dx$:

$$u(x) = -x, \quad du = -dx, \quad u(0) = 0, \quad u(-1) = 1.$$

We get

$$\int_0^{-1} f(x)\,dx = \int_0^1 f(-u)\,(-du) = \int_0^1 (-f(u))\,(-du) = \int_0^1 f(u)\,du.$$

So,

$$\int_1^{-1} f(x)\,dx = \int_1^0 f(x)\,dx + \int_0^{-1} f(x)\,dx = \int_1^0 f(x)\,dx + \int_0^1 f(x)\,dx = \int_1^1 f(x)\,dx = 0.$$

Remark. Using the same argument, one can show that for any odd function f Riemann integrable on $[-a,a]$, we have $\int_{-a}^{a} f(x)\,dx = 0$.

4. Consider the functions $f(x) = \sin(\frac{\pi}{2}x)$ and $g(x) = x$ on $[0,1]$.
   (a) Show that on $[0,1]$, the graph of f lies above the graph of g.
   (*Hint.* Consider the difference $h(x) = f(x) - g(x)$. To see that $h(x) \geq 0$ on $[0,1]$, find the minimum of h on $[0,1]$.)
   (b) Compute the area between the two graphs on $[0,1]$.

Solution. Consider the difference of the two functions, $h(x) = f(x) - g(x) = \sin(\frac{\pi}{2}x) - x$. To determine its minimal value, we need to compute its derivative:

$$h'(x) = \sin(\frac{\pi}{2}x)' - x' = \cos(\frac{\pi}{2}x)\frac{\pi}{2} - 1.$$

When x changes between 0 and 1, $\cos(\frac{\pi}{2}x)$ decreases from $\cos(0) = 1$ to $\cos(\frac{\pi}{2}) = 0$, so $h'(x)$ decreases from $\frac{\pi}{2} - 1$ (which is positive) to -1. This means that $h(x)$ increases up to a certain point (namely, $x = \frac{2}{\pi}\arccos(\frac{2}{\pi})$) and then decreases. Therefore, it takes its minimal value at one of its endpoints, 0 or 1. A direct computation yields

$$h(0) = \sin(0) - 0 = 0, \qquad h(1) = \sin(\frac{\pi}{2}) - 1 = 0.$$

Hence this minimal value is 0, and $h(x) \geq 0$ on $[0,1]$. In other words, $f(x) \geq g(x)$ on $[0,1]$, so the graph of f lies above the graph of g.
As a result, the area A between the two graphs is the difference between the area under the graph of f and area under the graph of g:

$$A = \int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx = \int_0^1 \sin(\frac{\pi}{2}x)\,dx - \int_0^1 x\,dx = \left[-\frac{2}{\pi}\cos(\frac{\pi}{2}x)\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$
$$= -\frac{2}{\pi}(\cos(\frac{\pi}{2}) - \cos(0)) - (\frac{1}{2} - 0) = \frac{2}{\pi} - \frac{1}{2}.$$

5. Compute $\int_0^1 x^3\,dx$ using two methods:
   (a) explicit Riemann sums, where you divide the interval $[0,1]$ into N equal parts and in each part choose as the point x_k^* the left endpoint;
   (b) the relation with the indefinite integral.
   Compare the answers obtained.

Solution.
   (a) According to the algorithm recalled in the Reminder, take a natural number N and divide the interval $[0,1]$ into N parts $[0,\frac{1}{N}]$, $[\frac{1}{N},\frac{2}{N}]$, …, $[\frac{N-1}{N},1]$ of equal size $\frac{1}{N}$. On each

subinterval $[\frac{k-1}{N},\frac{k}{N}]$, choose its left endpoint $\frac{k-1}{N}$. Then the corresponding Riemann sum for the function x^3 on $[0,1]$ becomes

$$0^3\cdot\frac{1}{N}+\left(\frac{1}{N}\right)^3\cdot\frac{1}{N}+\cdot+\left(\frac{N-1}{N}\right)^3\cdot\frac{1}{N}=\frac{1}{N^4}\sum_{k=1}^{N-1}k^3=\frac{1}{N^4}\frac{(N-1)^2N^2}{4}=\frac{1}{4}(1-\frac{1}{N})^2.$$

Here we applied the formula $\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$ for $n=N-1$. When N gets arbitrary large, $\frac{1}{N}$ approaches 0, so the numbers $\frac{1}{4}(1-\frac{1}{N})^2$ approach $\frac{1}{4}$. Thus $\int_0^1 x^3\,dx=\frac{1}{4}$.

(b) $\int_0^1 x^3\,dx=\left[\frac{x^4}{4}\right]_0^1=\frac{1}{4}(1^4-0^4)=\frac{1}{4}$.

The two methods for computing definite integrals yield the same result, but the second one demands much less efforts!