Homework/Tutorial 10

Marks: each question is worth 2 marks.

What this homework is about

You will learn how to compute definite integrals, and explore their relation with indefinite integrals and area computation.

Reminder

The **area** A under the graph of a continuous non-negative function f on an interval $[a,b]$ is defined as follows. For all integers N,

1) Divide $[a,b]$ into N subintervals $I_1,\dots I_N$ of equal size $\Delta_N = \frac{b-a}{N}$.
2) On each I_k, choose some point x_k^*.
3) On each I_k, approximate the figure under the graph of f by the rectangle $\Delta_N \times f(x_k^*)$.
4) Compute the total area $\sum_{k=1}^N f(x_k^*)\Delta_N$ of these small rectangles.

The area A is defined as the limiting value of these sums: $A = \lim\limits_{N\to+\infty} \sum\limits_{k=1}^{N} f(x_k^*)\Delta_N$.

The **definite integral** $\int_a^b f(x)\,dx$ of a function f on $[a,b]$ is the limit of similar sums (called **Riemann sums**), where the subintervals I_k are no longer of the same size, but become arbitrarily small: $\max\limits_k |I_k| \underset{N\to+\infty}{\longrightarrow} 0$. The function is called **(Riemann) integrable** on $[a,b]$ if this limit exists. For a continuous non-negative f, $\int_a^b f(x)\,dx$ computes the area under the graph of f on $[a,b]$.

The following formulas are useful for computing Riemann sums:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Definite integrals are usually evaluated using indefinite integrals, thanks to

The Fundamental Theorem of Calculus. Let f be continuous on $[a,b]$. Then:

Part 1: f is Riemann integrable on $[a,b]$, and $\displaystyle\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = F(b)-F(a)$, where F is any antiderivative of f on $[a,b]$;

Part 2: the function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f on $[a,b]$.

Properties of definite integrals:

- linearity: $\displaystyle\int_a^b (c_1 f_1(x)+\cdots+c_n f_n(x))\,dx = c_1\int_a^b f_1(x)\,dx+\cdots+c_n\int_a^b f_n(x)\,dx$;
- u-substitution: $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$;
- integration by parts: $\displaystyle\int_a^b f'(x)g(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b [f(x)g'(x)]\,dx$;
- $\displaystyle\int_a^a f(x)\,dx = 0$; $\qquad \displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$;
- additivity: $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$;
- monotony: $f(x)\ge g(x)$ on $[a,b]$ $\implies$ $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Questions

1. Consider the function $F(x) = \int_1^x \frac{dt}{\sqrt{3t+1}}$.
   (a) What is its natural domain?
   (b) Compute $F'(x)$.
   (c) Explain why F increases on its domain.
   (d) Compute $F(0)$.

2. Compute the following definite integrals:
   (a) $\int_1^0 x^2 \sqrt[5]{x^3-1}\,dx$;
   (b) $\int_1^{e^2} x \ln x\,dx$.

3. Compute
$$\int_1^{-1} f(x)\,dx, \quad \text{where } f(x) = \frac{\arctan(x)}{\sqrt{x^2+1}\cos(x)},$$
   without finding the antiderivative of f.
   (*Hint.* Compare $\int_1^0 f(x)\,dx$ and $\int_0^{-1} f(x)\,dx$.)

4. Consider the functions $f(x) = \sin(\frac{\pi}{2}x)$ and $g(x) = x$ on $[0,1]$.
   (a) Show that on $[0,1]$, the graph of f lies above the graph of g.
   (*Hint.* Consider the difference $h(x) = f(x) - g(x)$. To see that $h(x) \geq 0$ on $[0,1]$, find the minimum of h on $[0,1]$.)
   (b) Compute the area between the two graphs on $[0,1]$.

5. Compute $\int_0^1 x^3\,dx$ using two methods:
   (a) explicit Riemann sums, where you divide the interval $[0,1]$ into N equal parts and in each part choose as the point x_k^* the left endpoint;
   (b) the relation with the indefinite integral.
   Compare the answers obtained.