# Homework 1: Representations and characters of finite groups

**Instructions.** Try to give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those. The marks for the first 3 exercises sum up to 100; Exercise 4 gives you a bonus.

**Exercise 1.** Consider the group $G$ of invertible upper triangular matrices $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ over $\mathbb{C}$. Let $\rho\colon G \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ be the inclusion map.

1. Show that $(\mathbb{C}^2, \rho)$ is a representation of $G$.
2. What is the degree of $\rho$?
3. Prove that $\rho$ has precisely one sub-representation of degree 1.
4. Is this sub-representation isomorphic to a representation we have already seen?
5. Is $\rho$ irreducible?
6. Is $\rho$ indecomposable?
7. Under what condition is an indecomposable representation of a group necessarily irreducible? Does this condition hold for our group $G$?

**Exercise 2.** Consider the permutation representation $V_n^{perm} = \oplus_{i=1}^n \mathbb{C} e_i$ of the symmetric group $S_n$, with $S_n$ acting by $\sigma \cdot e_i = e_{\sigma(i)}$. Suppose $n \geq 3$.

1. Prove that $V_n^{perm}$ has precisely one sub-representation of degree 1. Denote it by $L$.
2. Identify the representation $L$.
3. Define a linear map $\varepsilon\colon V_n^{perm} \to \mathbb{C}$ by $\varepsilon(e_i) = 1$ for all $i$. Show that $V_n^{st} = \mathrm{Ker}\,\varepsilon$ is a sub-representation of $V_n^{perm}$. It is called the *standard representation* of $S_n$.
4. What is the degree of $V_n^{st}$?
5. Give a basis of $V_n^{st}$.
6. Verify that $V_n^{st}$ is irreducible.
7. Show that the $S_n$-representations $V_n^{perm}$ and $V^{tr} \oplus V_n^{st}$ are isomorphic. (Here $V^{tr}$ is the trivial degree 1 representation.)
8. Express the characters of $V_n^{perm}$ and $V_n^{st}$ in terms of fixed point sets for permutations.
9. A representation $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is called *faithful* if the map $\rho$ is injective. Which of our representations $V^{tr}$, $V_n^{perm}$, $V_n^{st}$ are faithful?

**Exercise 3.** In this exercise we will learn how to deform irreps for getting new ones.

Consider a finite group $G$ and its representations $(V, \rho)$ of degree $k$, and $(\mathbb{C}, \omega)$ of degree 1.

1. Verify that the map
$$G \to \mathrm{Aut}_{\mathbb{C}}(V),$$
$$g \mapsto \omega(g)\rho(g),$$
defines another $G$-representation on $V$. Denote it by $(V^\omega, \rho_\omega)$.
2. How are the characters $\chi^\rho$ and $\chi^{\rho_\omega}$ related?
3. When do these characters coincide?
4. Show that the $G$-representation $V$ is irreducible if and only if $V^\omega$ is so.
5. When are these two representations isomorphic?

**Example:** Specialise to the group $G = S_n$, $n \geq 4$, with the standard representation $(V_n^{st}, \rho)$ from Exercise 2. Put $\omega(\sigma) = \mathbf{sgn}(\sigma)$.

6. Verify that the signature **sgn** defines a degree 1 representation of $S_n$.
7. Show that $V_n^{st,\mathbf{sgn}}$ is an irreducible $S_n$-representation, not isomorphic to $V_n^{st}$.

**Bonus Exercise 4.** The aim of this exercise is to understand the set $\mathrm{Rep}_1(G)$ of degree 1 representations of a finite group $G$.

1. Show that a degree 1 representation is necessarily irreducible.
2. Prove that two such representations are isomorphic if and only if they coincide as maps from $G$ to $\mathrm{Mat}^*_{1\times 1}(\mathbb{C}) = \mathbb{C}^*$.

Denote by $[G, G]$ the set of all finite products of commutators $g_1 g_2 g_1^{-1} g_2^{-1}$, with $g_1, g_2 \in G$.

3. Show that $[G, G]$ is a normal subgroup of $G$.

$[G, G]$ is called the *commutator subgroup* of $G$. The quotient group $\mathrm{Ab}(G) = G/[G, G]$ is called the *abelianisation* of $G$. Consider the quotient map $\pi\colon G \to \mathrm{Ab}(G)$.

4. Verify that the quotient group $\mathrm{Ab}(G)$ is abelian.
5. How different can then $\mathrm{Rep}_1(\mathrm{Ab}(G))$ be from $\mathrm{Irrep}(\mathrm{Ab}(G))$?
6. Check that a degree 1 representation $\omega\colon G \to \mathbb{C}^*$ vanishes on $[G, G]$. It thus induces a map $\omega^*\colon\ \mathrm{Ab}(G) \to \mathbb{C}^*$.
7. Show that $\omega^*$ is a degree 1 representation of $\mathrm{Ab}(G)$.
8. Check that the maps
$$\begin{aligned}\mathrm{Rep}_1(G) &\leftrightarrow \mathrm{Rep}_1(\mathrm{Ab}(G)),\\ \omega &\mapsto \omega^*,\\ \theta\pi &\leftarrow\!\shortmid\, \theta\end{aligned}$$
are mutually inverse bijections.
9. **Example:** Using the presentation of the symmetric group $S_n$ in terms of the generators $\sigma_i$ (Lecture 1), obtain a group isomorphism $\mathrm{Ab}(S_n) \cong \mathbb{Z}_2$. Deduce from it a complete description of $\mathrm{Rep}_1(S_n)$.