A bridge between knotted graphs and axiomatizations of groups

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OCAMI, Osaka

FMSP Lectures
September 19, 2014

Based on:

✎ Qualgebras and Knotted 3-Valent Graphs, *to appear in Fundamenta Mathematica*

✎ (With S. Kamada) Alexander and Markov Theorems for Graph-Braids, *in progress*

A bridge between knotted graphs and axiomatizations of groups

topology

A bridge between knotted graphs and axiomatizations of groups

algebra

topology

Qualgebras: A bridge between knotted graphs and axiomatizations of groups

algebra

topology

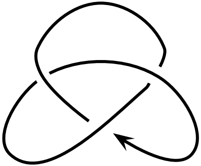
Part 1:

How a Knot Theorist Would Invent Qualgebras

Knot diagrams: from illustration to manipulation

1926 K. Reidemeister:

$$\text{Knots} \quad \cong \quad \text{Diagrams} \Big/ \text{R-moves}$$



Knot diagrams: from illustration to manipulation

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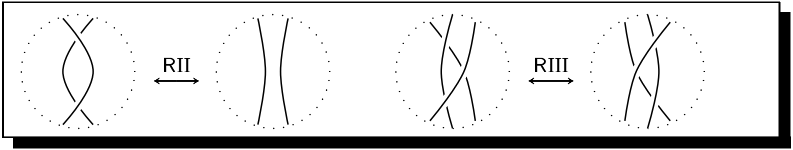
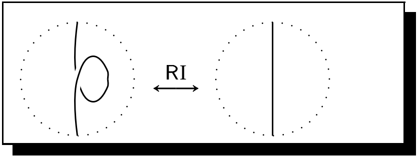
Reidemeister moves:





Knot diagrams: from illustration to manipulation

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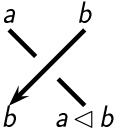
$$\text{Knots} \quad \cong \quad \text{Diagrams} \Big/ \text{R-moves}$$

Combinatorial knot invariants:

$$\begin{tikzcd}
& \text{Diagrams} \arrow[r, mapsto] \arrow[d, two heads] & \textcolor{green}{\text{something}} \\
\text{Knots} \quad \cong & \text{Diagrams} \Big/ \text{R-moves} \arrow[ur, dashed] &
\end{tikzcd}$$

Quandles as an algebraization of knots



colorings
by $(S, \lhd)$

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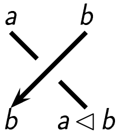

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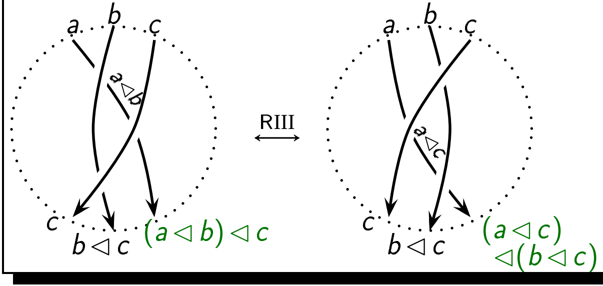

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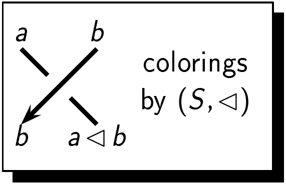
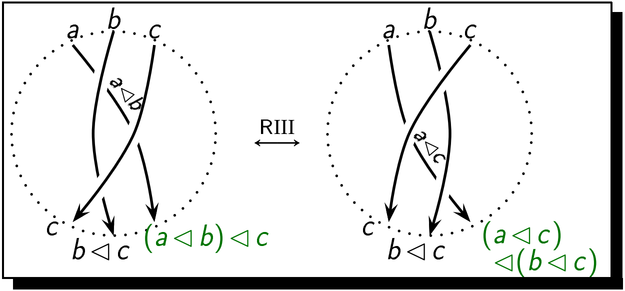
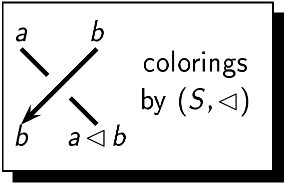


colorings by $(S, \triangleleft)$



$$\boxed{\text{RIII} \quad \leftrightarrow \quad (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \quad \text{(SD)}}$$

Quandles as an algebraization of knots



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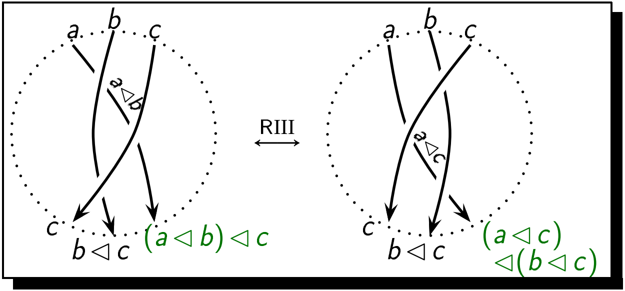

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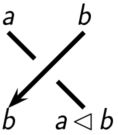
RII	↔	$(a \lhd b) \mathbin{\widetilde{\lhd}} b = a = (a \mathbin{\widetilde{\lhd}} b) \lhd b$	(Inv)
RI	↔	$a \lhd a = a$	(Idem)

Quandles as an algebraization of knots



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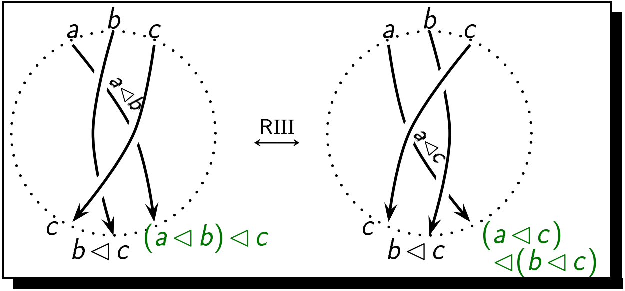

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(1982 D. Joyce,
S. Matveev)

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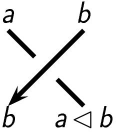

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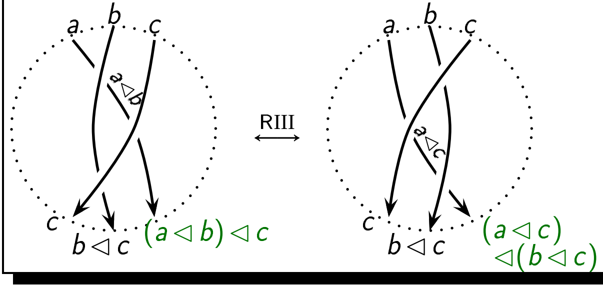

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knot invariants $\overset{\text{colorings}}{\leftsquigarrow}$ quandle

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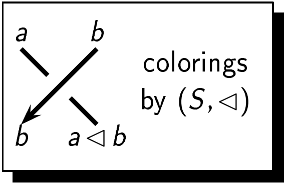

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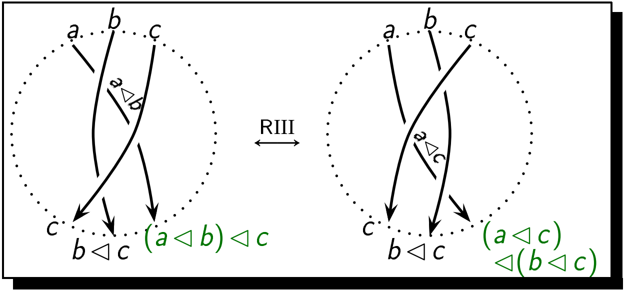

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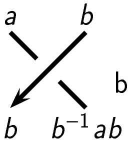
Conjugation quandles

Group G $\rightsquigarrow$ quandle $Conj(G)$:
$(G, g \lhd h = h^{-1}gh, g \mathbin{\widetilde{\lhd}} h = hgh^{-1})$.

Quandles as an algebraization of knots



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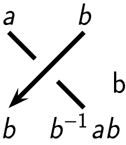
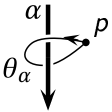


colorings by $Conj(G)$

Wirtinger presentation:



colorings by $Conj(G)$

$\updownarrow$

$Rep(\pi_1(\mathbb{R}^3 \setminus K), G)$

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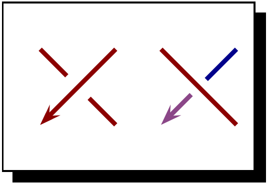
Quandles as an algebraization of knots



colorings by R_3

$\updownarrow$

Fox colorings

RIII	$\leftrightarrow$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	(SD)
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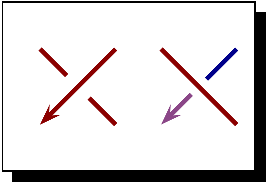
knot invariants $\overset{\text{colorings}}{\rightsquigarrow}$ quandle

Dihedral quandles

$R_3 = (\mathbb{Z}_3, a \lhd b = 2b - a).$

Quandles as an algebraization of knots



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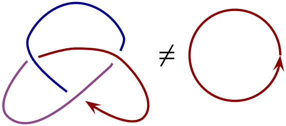

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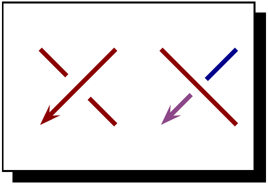
Quandles as an algebraization of knots



colorings by R_n

$\updownarrow$

Fox n-colorings

RIII	$\leftrightarrow$	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	(SD)
RII	$\leftrightarrow$	$(a \lhd b) \mathbin{\widetilde{\lhd}} b = a = (a \mathbin{\widetilde{\lhd}} b) \lhd b$	(Inv)
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Digression: The fundamental quandle of a knot

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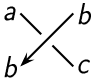
$$\text{Knots} \longrightarrow \text{Quandles}$$

$$K \longmapsto Q(K): \quad \text{generators} \leftrightarrow \text{arcs of } D_K$$

$$\text{relations} \leftrightarrow \text{crossings of } D_K$$

$$\boxed{c = a \lhd b}$$



⚠ Does not depend on the choice of a diagram D_K of K.

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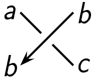

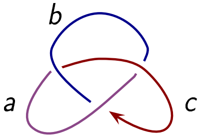

$$a = c \triangleleft b$$
$$b = a \triangleleft c$$
$$c = b \triangleleft a$$

Digression: The fundamental quandle of a knot

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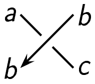

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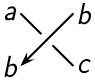

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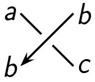

Conclusion: The fundamental quandle is a universal weak knot invariant.
Problem: Quandles are hard to compare.

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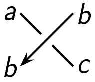

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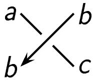

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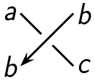

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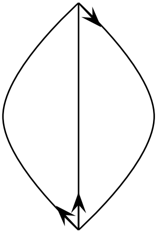
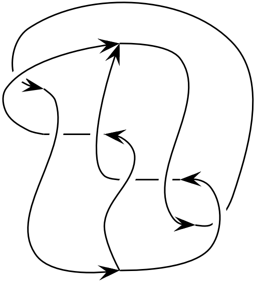
Solution: Compare $Rep_{Qu}(Q(K), Q)$ for "simple" Q $\longleftrightarrow$ colorings by Q.
Consequence: Quandle invariants are very powerful.

Knotted 3-valent graphs

Standard and Kinoshita-Terasaka Θ***-curves***:



$\boxed{\Theta_{st}}$



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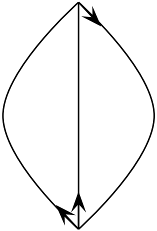
Knotted 3-valent graphs

Standard and Kinoshita-Terasaka Θ-curves:



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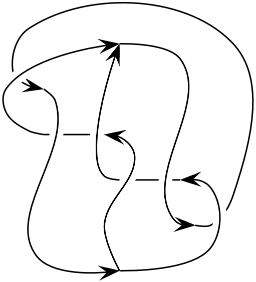

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Importance:

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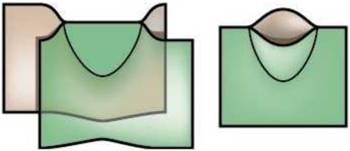

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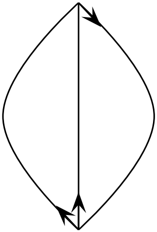

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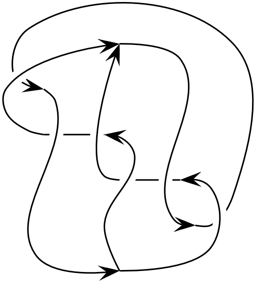

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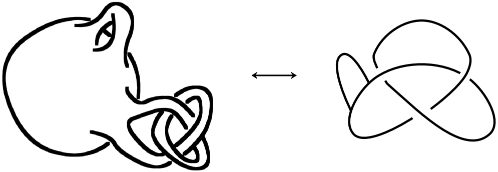

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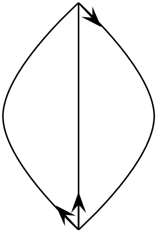

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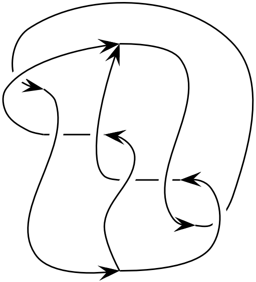

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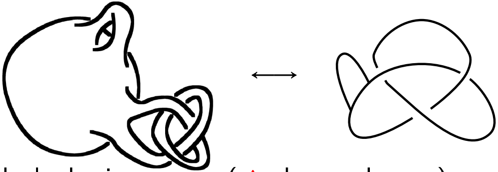
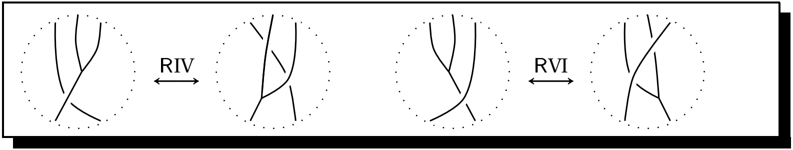
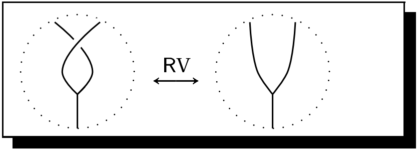


- form a finitely presented algebraic system (⚠ knots do not).

Reidemeister moves for 3-graphs

1989 L.H. Kauffman, S. Yamada, D.N. Yetter:





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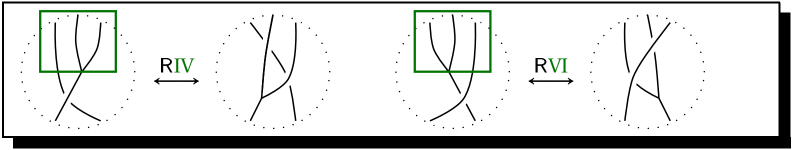

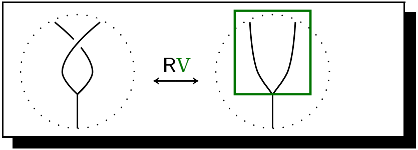

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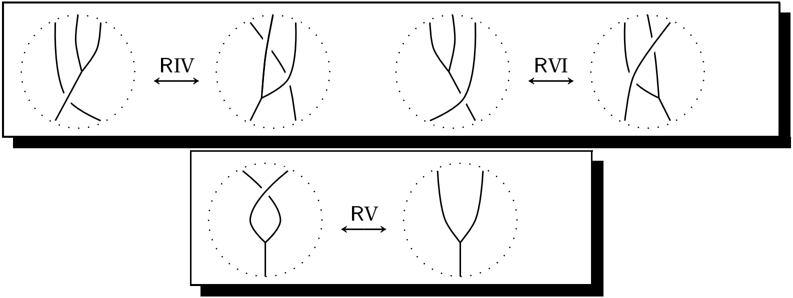

3-Graphs $\cong$ $\left. \text{Diagrams} \middle/ \text{RI-RVI} \right.$

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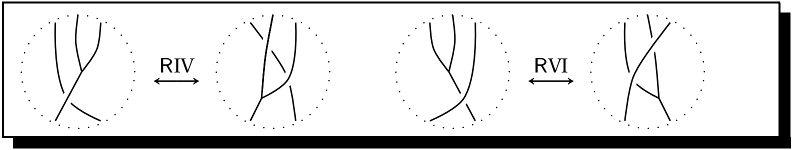

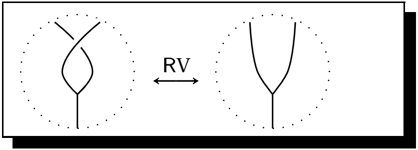

Combinatorial graph invariants:

$$\begin{tikzcd} & & \text{Diagrams} \arrow[r, mapsto] \arrow[d, two heads] & \text{something} \\ \text{3-Graphs} \arrow[r, phantom, "\cong"] & {} & \text{Diagrams} \,/\, \text{RI-RVI} \arrow[ur, dashed] & \end{tikzcd}$$

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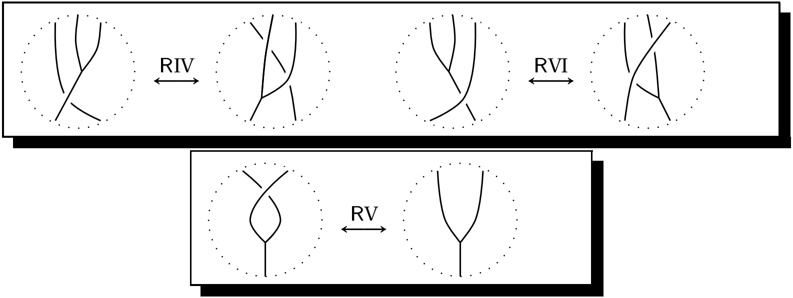

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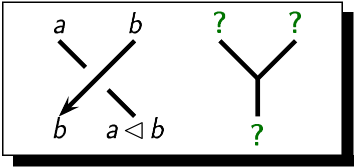
Question: Can "something" be related to quandles?

Quandle colorings for 3-graphs

A more precise question: How to extend quandle colorings to 3-graphs?



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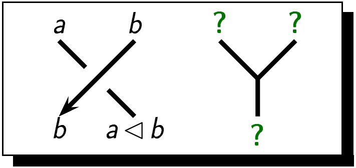
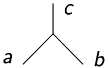


Answers:

- 

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$a^{\pm1}b^{\pm1}c^{\pm1}=1$

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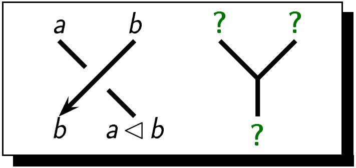
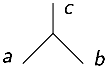


Answers:

❀



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- *G-family of quandles* (2012 Ishii-Iwakiri-Jang-Oshiro);
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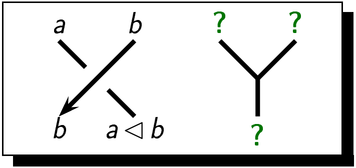

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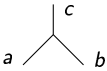

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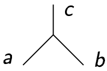

S is a quandle; $\lhd^{+} = \lhd$, $\lhd^{-} = \widetilde{\lhd}$
$\forall x, ((x \lhd^{\pm} a) \lhd^{\pm} b) \lhd^{\pm} c = x$

(2010 M. Niebrzydowski)

Quandle colorings for 3-graphs

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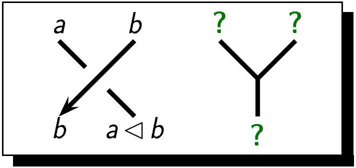

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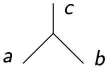
- 

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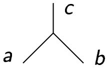
- 

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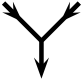
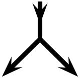
- And some more...

Orientation

Poleless 3-graphs: only ***zip***  and ***unzip***  vertices.

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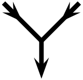
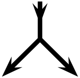
Poleless 3-graphs: only ***zip***  and ***unzip***  vertices.

Proposition: Every 3-graph admits a poleless orientation.

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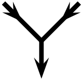
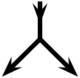
Poleless 3-graphs: only ***zip***  and ***unzip***  vertices.

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Unoriented 3-graph $\longmapsto$ { its poleless orientations }.

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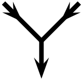
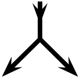
Poleless 3-graphs: only ***zip***  and ***unzip***  vertices.

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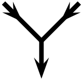
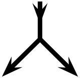
Poleless 3-graphs: only ***zip***  and ***unzip***  vertices.

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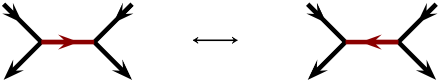
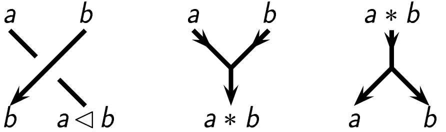
Theorem (A. Ishii): All poleless orientations of a given 3-graph are connected by single-edge orientation switches:



Qualgebras as an algebraization of 3-graphs

colorings
by $(S, \lhd, *)$



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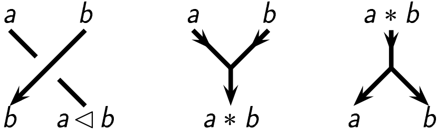

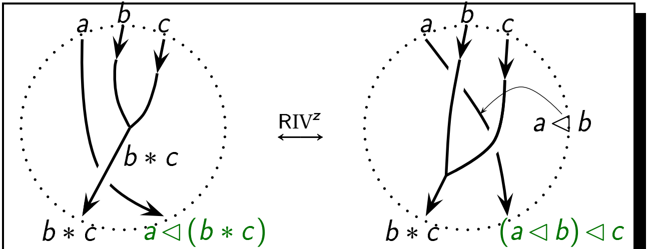

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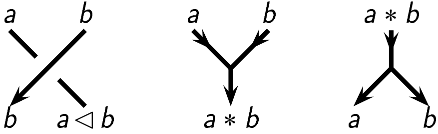

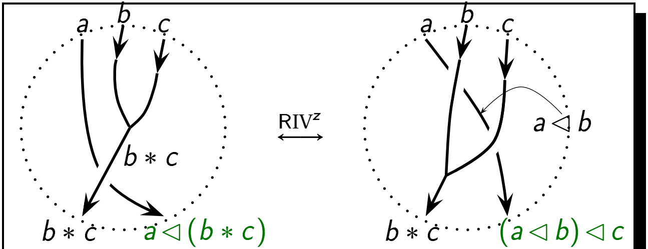

RIV $\leftrightarrow$ $a \lhd (b * c) = (a \lhd b) \lhd c$
RVI $\leftrightarrow$ $(a * b) \lhd c = (a * c) \lhd (b * c)$
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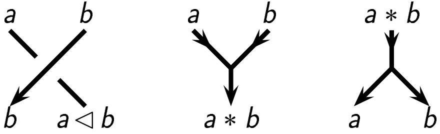

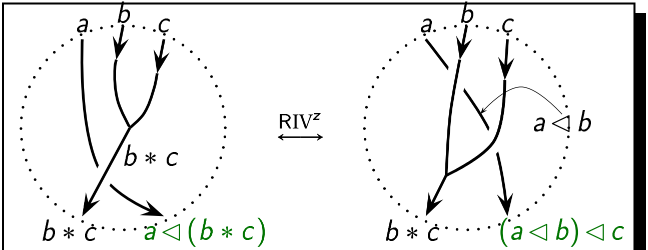

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& (SD),
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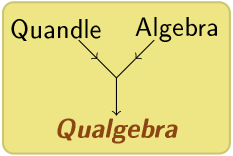

Quandle Algebra

Qualgebra

Qualgebras as an algebraization of 3-graphs

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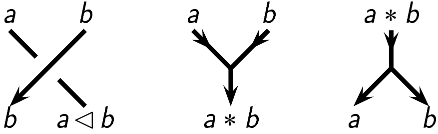

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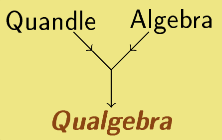
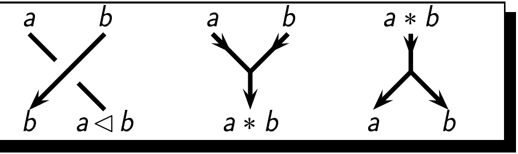

Quandle Algebra

Qualgebra

3-graph invariants $\overset{\text{colorings}}{\leftsquigarrow}$ qualgebra

Qualgebras as an algebraization of 3-graphs

colorings by $(S, \lhd, *)$



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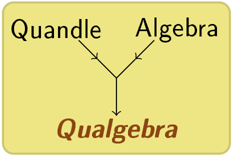


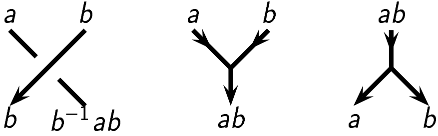
3-graph invariants $\overset{\text{colorings}}{\leftsquigarrow}$ qualgebra

Group qualgebras

Group $G \rightsquigarrow$ qualgebra $QA(G) = (G, g \lhd h = h^{-1}gh, g * h = gh)$.

Qualgebras as an algebraization of 3-graphs

colorings by $QA(G)$



$$\text{3-graph invariants} \overset{\text{colorings}}{\leftsquigarrow} \text{qualgebra}$$

Group qualgebras

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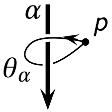
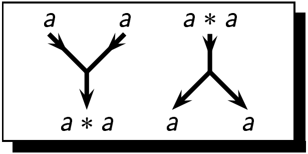
Wirtinger presentation:



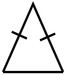
$$\text{colorings by } QA(G) \updownarrow Rep(\pi_1(\mathbb{R}^3 \setminus \Gamma), G)$$

Computation example

Isosceles colorings:



Linguistic remark:



Isosceles triangle

Computation example

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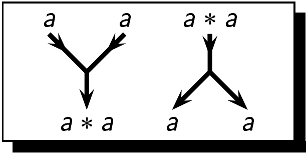

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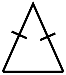

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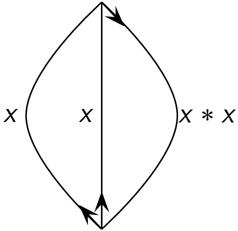
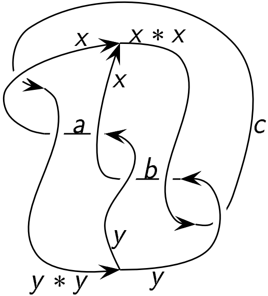
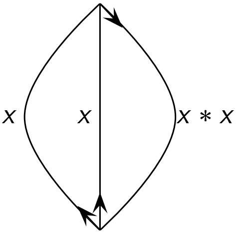


$\boxed{\Theta_{st}}$



$\boxed{\Theta_{KT}}$

Computation example



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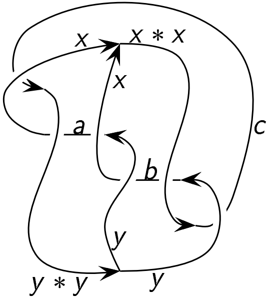

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$$(\star)\begin{cases} a &= x \triangleleft (y * y) = y \triangleleft x, \\ b &= x \mathbin{\widetilde{\triangleleft}} y = y \mathbin{\widetilde{\triangleleft}} (x * x), \\ c &= (y * y) \triangleleft x = (x * x) \mathbin{\widetilde{\triangleleft}} y. \end{cases}$$

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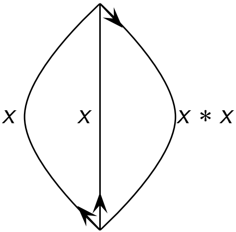

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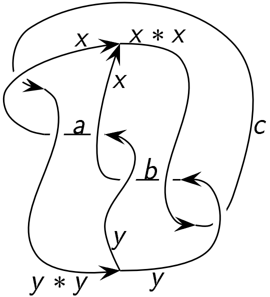

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Group qualgebra $(S_4, g \triangleleft h = h^{-1}gh, g * h = gh)$.

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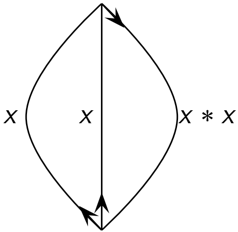

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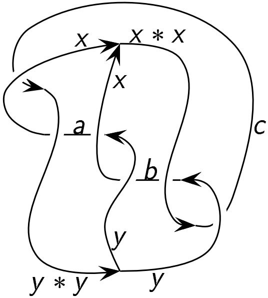

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Solutions: $x = y$ and $x = (123), y = (432)$ and ...

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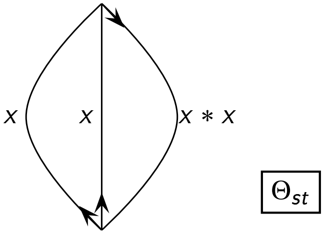

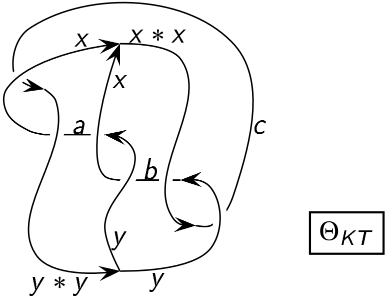

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$$\#Col^{iso}_{S_4}(\Theta_{st}) = \#S_4 \qquad \#Col^{iso}_{S_4}(\Theta_{KT}) \geqslant \#S_4 + 1$$

Part 2:

How an Algebraist Would Invent Qualgebras

Group qualgebras

Example 1

Group $G \rightsquigarrow$ ***conjugation quandle*** $Conj(G) = (G, g \lhd h = h^{-1}gh)$.

1982 D. Joyce: Quandle axioms $\iff$ all properties of conjugation.

Group qualgebras

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Example: In any $QA(G)$,

$$g \lhd (h \lhd k) = ((g \mathbin{\widetilde{\lhd}} k) \lhd h) \lhd k$$

$$((k^{-1}hk)^{-1}gk^{-1}hk = k^{-1}h^{-1}kgk^{-1}hk)$$

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$\Rightarrow$ the equation follows from quandle axioms:

$$g \lhd (h \lhd k) \overset{(Inv)}{=} ((g \mathbin{\widetilde{\lhd}} k) \lhd k) \lhd (h \lhd k) \overset{(SD)}{=} ((g \mathbin{\widetilde{\lhd}} k) \lhd h) \lhd k.$$

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abstract level	quandle axioms	
topology	moves RI-RIII	
groups	conjugation	

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⚠ Qualgebra axioms $\not\Rightarrow$ all relations btwn conjugation & multiplication.

Example: $(b \lhd a) * (a \lhd b) = ((a \mathbin{\widetilde{\lhd}} b) \lhd a) * b$

in $QA(G)$: $a^{-1}bab^{-1}ab = a^{-1}bab^{-1}ab$

in free qualgebras: false

Other qualgebra examples

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Example 1′

Group G & $X \subset G \rightsquigarrow$ the sub-qualgebra of $QA(G)$ generated by X.

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$\rightsquigarrow$ Abstract graph invariants.

An infinite family of exotic examples

Example S_n

Set X & commutative operation $\star$ & zero element 0 ($0 \star x = x \star 0 = 0$)

An infinite family of exotic examples

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$$Q_{X,n} = \{((x_1,\ldots,x_n),g) \in X^{\times n} \times S_n \,|\, x_i = x_j = 0 \text{ whenever } g(i) = j \neq i\},$$

$$(\overline{x},g) \lhd (\overline{y},h) = (\overline{x}\cdot h, h^{-1}gh),$$

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$\rightsquigarrow$ two 5-element qualgebras.

- Commutative.
- Associative.
- Not cancellative $\Longrightarrow$ do not come from groups:

$$((0,0),\tau) *_i ((x_1,x_2),\mathrm{Id}) = ((0,0),\tau), \qquad i = 1,2.$$

Towards a classification

Example 3

Size $\leqslant 3$: only trivial qualgebras.

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Non-trivial qualgebra structures on $Q = \{p, q, r, s\}$:

$$\text{put } \overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s;$$

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$\rightsquigarrow \quad 3 * 3 = 9$ structures.

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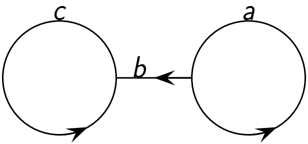
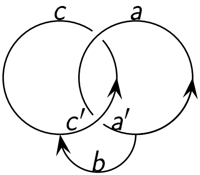
Question: Continue the classification.

Computation example

Standard and Hopf cuff graphs:



$\boxed{C_{st}}$



$\boxed{C_H}$

$$a' = ((a \mathbin{\widetilde{\lhd}} c) \lhd a$$
$$c' = c \lhd a$$

$$\#\mathcal{C}ol_Q(C_{st}) = \#\{(a,b,c) \in Q \,|\, b * a = a,\ b * c = c\} = 18,$$
$$\#\mathcal{C}ol_Q(C_H) = \#\{(a,b,c) \in Q \,|\, b * a = a \lhd c,\ b * c = c \lhd a\} = 14.$$

Qualgebras with inversion

Question: How far qualgebras are from groups?

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Step 1: inversion & cancellativity.

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Good involution: $\rho : S \to S$ s.t.

$$\left.\boxed{\begin{array}{l} \rho(\rho(a)) = a \\ \rho(a) \lhd b = \rho(a \lhd b) \\ a \lhd \rho(b) = a \mathbin{\widetilde{\lhd}} b \end{array}}\right\}$$

Symmetric quandle (1996 S. Kamada)

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Group qualgebra is symmetric

Group $G \rightsquigarrow QA^*(G) = (G, g \lhd h = h^{-1}gh, g * h = gh, \rho(h) = h^{-1})$.

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Properties:

- Maps $a \mapsto a * b$ and $a \mapsto b * a$ are bijections
  $\rightsquigarrow$ $*$ is a Latin square (= pseudo-sudoku).

Qualgebras with inversion

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Group qualgebra is symmetric

Group $G \rightsquigarrow QA^*(G)=(G, g\lhd h=h^{-1}gh, g*h=gh, \rho(h)=h^{-1})$.

Properties:

- Maps $a \mapsto a*b$ and $a \mapsto b*a$ are bijections
  $\rightsquigarrow$ $*$ is a Latin square (= pseudo-sudoku).
- ρ is defined uniquely.

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Question: When are qualgebra invariants independent of orientations?

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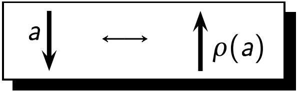

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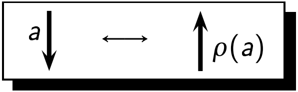

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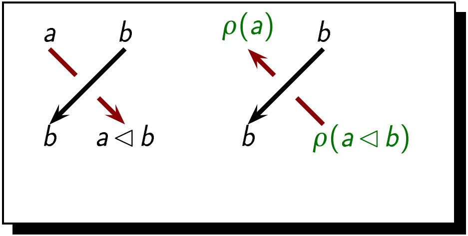

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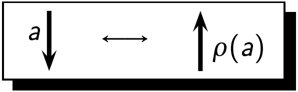

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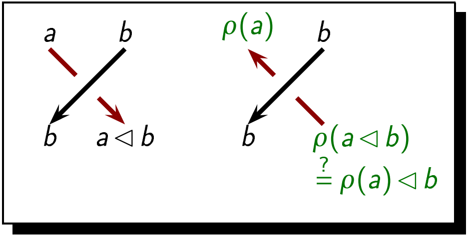

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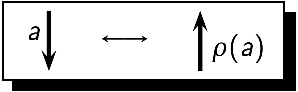

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The first three equations are braced together with ***Symmetric quandle***, and all four with ***Symmetric qualgebra***.

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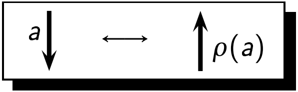

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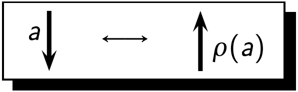

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abstract level	good involution axioms for qualgebras
topology	unoriented 3-graphs
groups	conjugation- and multiplication-inversion interactions

Symmetric qualgebras: examples

Example 0

Symmetric trivial qualgebras $\longleftrightarrow$ Latin squares which

- are symmetric w.r.t. the main diagonal, and
- contain a row $\sigma \in Bij(S)$ iff contain a row σ^{-1}.

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Example 3

$S = \{x, y, z\}$, $a \lhd b = a$, $*$ is commutative.

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y
ρ	x	z	y

$QA(\mathbb{Z}_3)$

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not groups

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- Trivial qualgebras:

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z	z	w	x	y
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ρ	x	w	z	y

$\leftarrow QA(\mathbb{Z}_4)$

$QA(\mathbb{Z}_2 \times \mathbb{Z}_2) \rightarrow$

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Question: How far qualgebras are from groups?

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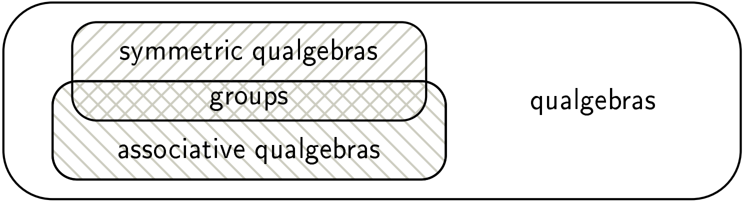

From quandles to qualgebras: theory

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That is, $T_{b \lhd c} = T_c^{-1} T_b T_c$.

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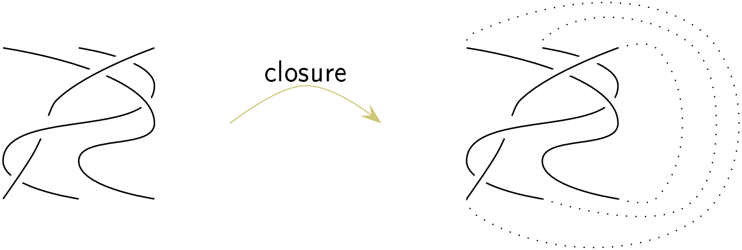
Part 3:

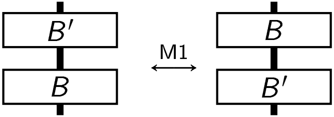
Variations of Qualgebra Ideas

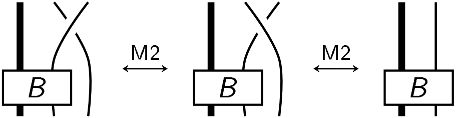
Alexander-Markov theorem



Theorem (1923 Alexander; 1935 Markov)

- Surjectivity.
- Kernel: moves M1 and M2.





Braid and knot invariants

Knot invariants out of braid invariants:

$$\begin{tikzcd}
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Many knot invariants adapt to the braid case, with possible enhancements.

Example: quandle invariants.

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Combinatorial braid invariants:

$$\begin{tikzcd} & \text{Braid diagrams} \arrow[r, mapsto] \arrow[d, two heads] & \text{something} \\ \text{Braids} \cong & \text{Diagrams} \Big/ \text{RII-RIII} \arrow[ur, dashed] & \end{tikzcd}$$

Many knot invariants adapt to the braid case, with possible enhancements.

Example: quandle invariants.

- Weaker structure: ***rack*** (= quandle without $a \lhd a = a$) $\Longleftarrow$ no RI.

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- Weaker structure: ***rack*** (= quandle without $a \lhd a = a$) $\Longleftarrow$ no RI.
- Operator invariants instead of counting invariants:

$$\begin{aligned} \text{braids with } n \text{ strands } &\longrightarrow \mathrm{Aut}(S^{\times n}), \\ \beta &\longmapsto (\text{ colors of upper arcs} \\ &\qquad\quad \mapsto \text{ colors of lower arcs }). \end{aligned}$$

Branched Alexander-Markov theorem

Question: : A closure procedure for 3-graphs?

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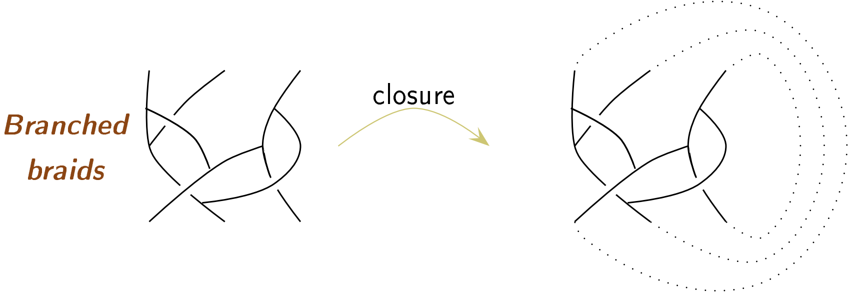

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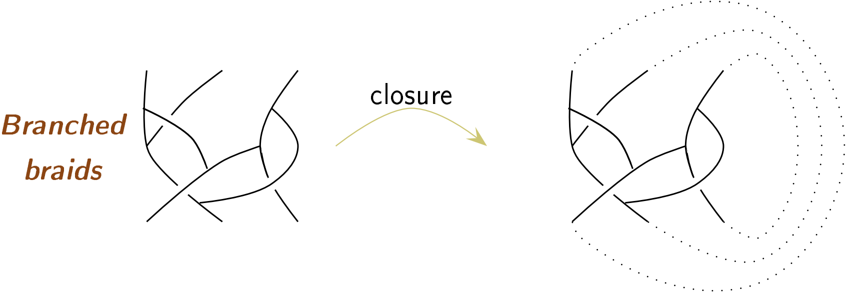

Theorem (2010 K. Kanno - K. Taniyama; 2014 S. Kamada - L.)

- Surjectivity.
- Kernel: moves M1 and M2.

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$$\text{branched braids} \xrightarrow{\text{closure}} \text{3-graphs}$$

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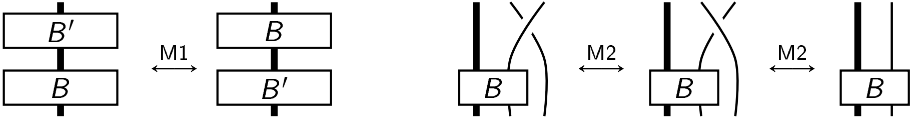

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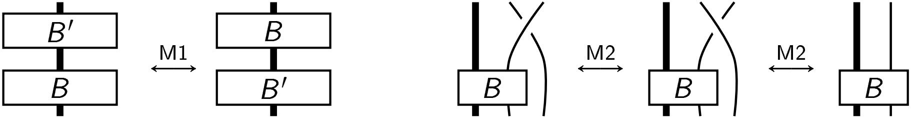

Generalizations

- Graph-braids (vertices of arbitrary valence).
- Virtual and welded versions.

Branched braid and 3-graph invariants

3-Graph invariants out of B-braid invariants:

$$\begin{tikzcd}
 & B\text{-braids} \arrow[r, mapsto] \arrow[d, two heads] & \text{something} \\
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B-braid invariants $\overset{\text{colorings}}{\rightsquigarrow}$ ***weak qualgebra*** (omit $a \lhd a = a$)

Getting more out of qualgebra colorings

diagrams: $D \overset{\text{R-move}}{\rightsquigarrow} D'$

colorings: $\mathscr{C} \rightsquigarrow \mathscr{C}'$

coloring sets: $\mathcal{C}ol_S(D) \overset{\text{bij}}{\longleftrightarrow} \mathcal{C}ol_S(D')$

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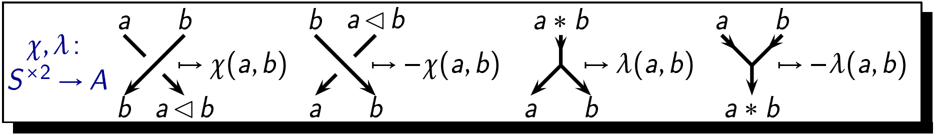
Inspiration: ***Quandle cocycle invariants*** of knots (1999 Carter-Jelsovsky-Kamada-Langford-Saito).

Qualgebra cocycle invariants for 3-graphs



$\chi, \lambda \colon S^{\times 2} \to A$

$\mapsto \chi(a,b)$ $\quad \mapsto -\chi(a,b)$ $\quad \mapsto \lambda(a,b)$ $\quad \mapsto -\lambda(a,b)$

$$\text{coloring } \mathscr{C} \longmapsto \omega(\mathscr{C}) = \sum_{\text{crossings}} \pm\chi(a,b) \quad + \sum_{\text{vertices}} \pm\lambda(a,b)$$

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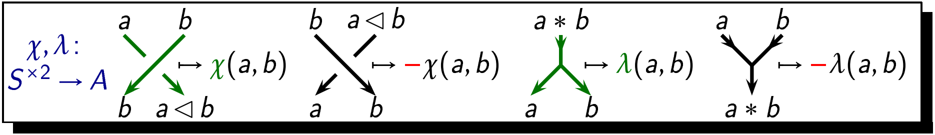

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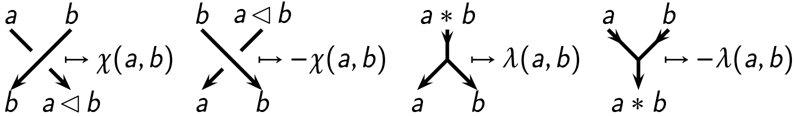

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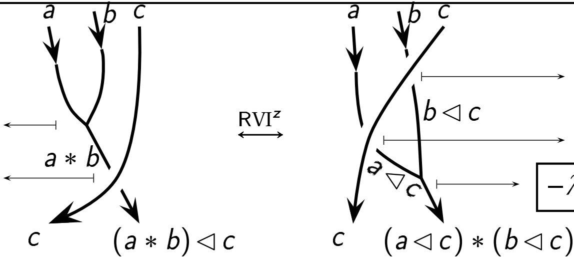

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$\boxed{\chi(b,c)}$

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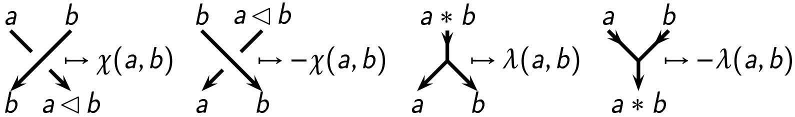

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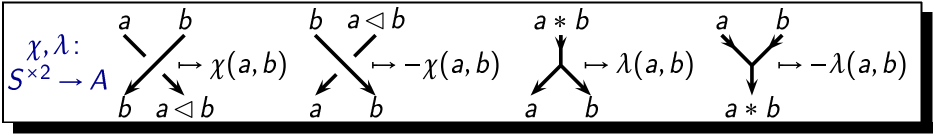

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RI-RIII are automatic.

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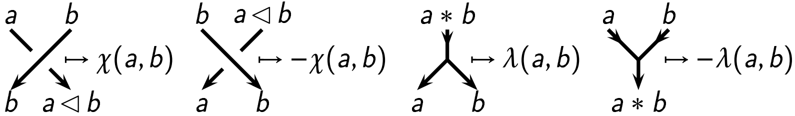

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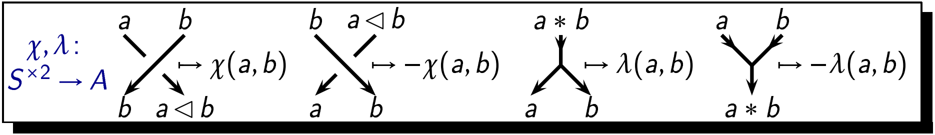

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Qualgebra cocycle invariants $\supseteq$ qualgebra counting invariants

$(\leftsquigarrow \chi = \lambda = 0)$.

Towards qualgebra cohomology

Qualgebra 2-***coboundaries*** ($\subseteq$ qualgebra 2-cocycles):

$$\phi : S \to A \quad \rightsquigarrow \quad \begin{array}{l} \chi(a,b) = \phi(a \lhd b) - \phi(a), \\ \lambda(a,b) = \phi(a) + \phi(b) - \phi(a * b) \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \text{trivial} \\ \text{graph invariants} \end{array}$$

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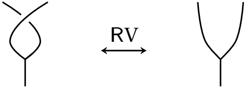

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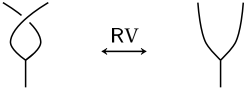

Enhancements

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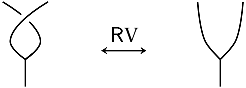

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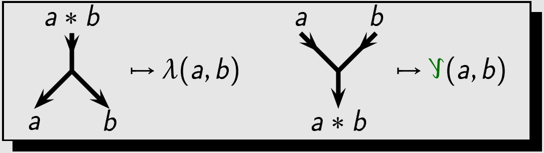
- Distinguish zip- and unzip-vertices:



$\mapsto \lambda(a,b)$ $\quad$ $\mapsto \Upsilon(a,b)$

Qualgebra cocycles: example

Example 4

$$A = \mathbb{Z}, \quad Q = \{p, q, r, s\} \qquad \overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s;$$

$$x \lhd r = \overline{x}, \qquad x \lhd y = x \text{ for other } y;$$

$$* \text{ is commutative}, \qquad \overline{x} * \overline{y} = \overline{x * y},$$

$$r * x = r \text{ for } x \neq r,$$

$$r * r = s * s = p * q = s, \qquad p * p, p * s \in \{p, q, s\}$$

- $Z^2(Q) \cong \mathbb{Z}^8$
- $B^2(Q) \cong \mathbb{Z}^4$
- $H^2(Q) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^4$

どうもありがとう