A billiard containing all links

Un billard réalisant tout entrelacs

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A B S T R A C T

We construct a 3-dimensional billiard realizing all links as collections of isotopy classes of periodic orbits. For every branched surface supporting a semi-flow, we construct a 3d-billiard whose collections of periodic orbits contain those of the branched surface. R. Ghrist constructed a knot-holder containing any link as collection of periodic orbits. Applying our construction to his example provides the desired billiard.

RÉSUMÉ


For every compact domain of $\mathbb{R}^3$ with a smooth boundary, one can play billiard inside it, with the rule that rays reflect perfectly on the boundary, see [6]. If the boundary has corners, the reflection is not defined, and we only consider orbits avoiding them. Thus a periodic orbit with no self-intersection point yields a knot in $\mathbb{R}^3$ and one can wonder about the relation between the shape of the billiard and the knots arising in this way. In a cube, the latter correspond to the so-called Lissajous knots, see [2]. In a cylinder the situation is more intricate, see [3]. It is asked in [4] and [5] whether there exists a billiard containing all knots as periodic orbits. In this Note, we provide a positive answer to this question.

Definition 1. A template (see Figs. 1, 2) is a smooth compact surface $S$ with boundary embedded in $\mathbb{R}^3$ and equipped with a non-vanishing vector field $V$ so that:

(i) $V$ is tangent to the surface $S$ and to its boundary,

(ii) there exist finitely many branching segments — called convergence segments — transverse to $V$, where three pages $P^+, P^-$ and $P^0$ of the surface meet, with $V$ leaving $P^+$ and $P^-$ and entering $P^0$,

(iii) there exist finitely many branching points — called separation points — on the boundary of $S$ whose neighborhood is diffeomorphic to an open disc cut along the bottom vertical radius and equipped with the top–bottom vector field.

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First suppose we are given a template $T_0$ that can be immersed into the plane (for example Ghrist’s template works, but not the generalized Lorenz’).\footnote{For orientable templates, the so-called bell trick can do the job.} Distort it into $T_1$ in such a way that the projection of $T_1$ on the horizontal plane is obtained by gluing ribbons with slopes $(\pm 1, 0)$ or $(0, \pm 1)$ for generic points, isosceles-rectangular triangles for changes of direction, ribbons with slope $(0, -1)$ for convergence segments, and parabolic church shapes. These patterns are depicted in Figs. 3 and 4. Note that they fit well into the integer lattice $\mathbb{Z}^2$. Now we associate a billiard $B_1$ to $T_1$ by lifting it in $\mathbb{R}^3$ so that ribbons are 1 unit thick along the vertical direction and match with each other. Convergence segments and separation points deserve a special treatment depicted in Figs. 5 and 6. There could be level gaps, but these can be settled using vertical double bends, see Fig. 7.

Our claim is the following:

**Theorem 2.** For every template $T_0$ embedded in $\mathbb{R}^3$, every finite collection of periodic orbits of $T_0$ is isotopic to a finite collection of periodic orbits of the billiard $B_1$ constructed above.

Applying this to Ghrist’s template [1] yields

**Corollary 3.** There exists a domain in $\mathbb{R}^3$ with a piecewise smooth boundary (see Fig. 9) so that any link appears as a family of periodic billiard trajectories.

**Proof.** We only prove the theorem for knots, the case of links being similar.

Let $\gamma_0$ be a periodic orbit on $T_0$. Since the templates $T_0$ and $T_1$ are isotopic, there exists a periodic orbit $\gamma_1$ of $T_1$ isotopic to $\gamma_0$. Let $p$ be an arbitrary point on $\gamma_1$. One associates an infinite periodic word $w_{\gamma_1,p}^\mathbb{N}$ on the alphabet $\{0, 1\}$ so that when one follows $\gamma_1$, the sequence of left/right-choices at separation points is described by the letters of $w_{\gamma_1,p}^\mathbb{N}$.

Let $q$ be a point inside $B_1$, and suppose that it projects to $T_1$ on a point where the flow is parallel to the $y$-direction. Call $(x_q, y_q)$ the horizontal coordinates of $q$ and $z_q$ the vertical one (along which the projection is performed). Playing billiard in $B_1$ along the $y$-direction does not change $(x_q, z_q)$, unless one crosses corners, churches or convergence boxes. In the first
Fig. 5. How to realize a separation point using billiards: on the left side a horizontal cut. The left two curves are confocal parabolae, the exterior one being homothetic to the interior one by a factor 2. In this way, vertical entering rays go out vertically and their mutual distances are doubled. The same thing holds on the right of the separation point.

Fig. 5. Réalisation d’un point de séparation par un billard. Les deux courbes de gauche sont des paraboles confocales, l’externe étant homothétique de l’intérieur par un facteur 2. Ainsi les trajectoires arrivant verticalement par le haut sortent verticalement et leurs distances mutuelles sont doublées. La même chose se produit à droite du segment de séparation.

Fig. 6. How to realize a convergence segment by billiards: a vertical cut. The shape is the same as for separation points, but the flow is reversed, and the shape turned 90° along the y-direction.

Fig. 6. Réalisation d’un segment de convergence. On prend la même pièce que pour les points de séparation, en la tournant verticalement. Le flot est également renversé.

Fig. 7. A horizontal double bend. At each corner some crossings which were not on the template $T_0$ may appear. Nevertheless, since corners come in pairs, these extra-crossings disappear with a Reidemeister-II-move. Level gaps are solved with this pattern turned vertically.

Fig. 7. Une chicane horizontale. À chaque coin apparaissent des croisements qui n’existaient pas sur le patron $T_0$. Néanmoins, comme les coins viennent toujours par paires, ces croisements supplémentaires aussi, et un mouvement de Reidemeister les supprime.

Fig. 8. How to realize a half-twist. Note on the left picture that the projections of any two strands cross exactly once, and that horizontal strands go above vertical ones. Since a braid in which any two strands cross exactly once and positively is half-turn $\Delta_1/\Delta_n$, this billiard realizes a half-twist.

Fig. 8. Réalisation d’un demi-tour. Comme indiqué sur la partie gauche, les projections horizontales de deux trajectoires se coupent exactement une fois, avec le brin O–E par au-dessus du brin N–S. Or une tresse dont deux brins quelconques se coupent exactement une fois et positivement est isotope au demi-tour $\Delta_n$.

It is therefore natural to compare the orbit $\gamma_1$ on $T_1$ passing through $p$ and the orbit $\tilde{\gamma}_1'$ on $B_1$ passing through $(0, w^{(N)}_{T_1}, p)$ and going along the $y$-direction. Since $\tilde{\gamma}_1'$ is horizontal except in convergence boxes, it is the lift of a periodic orbit $\gamma_1'$ in the planar template $T_1$ considered as a planar billiard. Therefore a knot-diagram of $\tilde{\gamma}_1'$ is obtained from $\gamma_1'$ by removing the ambiguities at crossings.

A crossing of $\gamma_1'$ may arise in three situations only. Either it arises in a corner with two strands in the same box, in which case the previous-or-next corner provides another crossing for the same pair of strands, so that the pair will disappear with a Reidemeister-II-move, see Fig. 7. Or it arises when different ribbons cross, in which case the same ribbons cross in $T_1$. Or it arises at a corner when two ribbons become parallel just before a convergence box, in which case the crossing already exists in $\gamma_1$ when the two ribbons of $T_1$ overlap. Therefore, the horizontal projection of $\tilde{\gamma}_1'$ can be distorted to $\gamma_1'$ using Reidemeister-II-moves only, and so the two knots are isotopic.

We still have to address the case of a half-twist on a non-orientable template. This can be fixed with the billiard of Fig. 8. So the proof is complete. $\square$
Note that our construction can be smoothed so that the boundary of the billiard becomes a smooth surface. On the other hand, the parabolae are crucial in order to double the coordinates, preventing us to construct a billiard with piecewise-linear boundary. We are left with these two questions:

(i) Is it possible to construct a polygonal billiard containing all links as periodic orbits?
(ii) Is it possible to construct a convex billiard containing all links as periodic orbits?

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References