Supplementary problem sheet 3

Integral calculus

1. Compute $\int (3x-1)^2\,dx$ using two methods:
   (a) first compute the square, then integrate term by term;
   (b) use the u-substitution $u = 3x - 1$.
   Compare the two results. What explains their difference?

2. Compute the following indefinite integrals:
   (a) $\int \cos^2(x)\,dx$;
   (b) $\int \cos^3(x)\,dx$;
   (c) $\int \tan^4(x)\,dx$;
   (d) $\int \dfrac{(x+1)^2}{x^2+4}\,dx$;
   (e) $\int \dfrac{1}{x^2-1}\,dx$;
   (f) $\int \dfrac{1}{x(x^2+1)}\,dx$.

3. Compute $\displaystyle\int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x))\,dx$.

4. Let n be a positive integer. Compute $\displaystyle\int_0^1 x(1-x)^n\,dx$.

5. Let n and m be positive integers. Show that
$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx.$$
   You do not need to compute these integrals.

6. Given an odd function f integrable on $[-a, a]$, show that $\displaystyle\int_{-a}^a f(x)\,dx = 0$.

7. Find the following limits by evaluating appropriate definite integrals over $[0, 1]$:
   (a) $\displaystyle\lim_{N\to+\infty} \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{N}}{N^{\frac{3}{2}}}$;
   (b) $\displaystyle\lim_{N\to+\infty} \sum_{k=1}^{N} \frac{1}{N}\cos\left(\frac{k\pi}{2N}\right)$.

8. Evaluate the following integrals using geometric arguments:
   (a) $\displaystyle\int_{-4}^4 \sqrt{16-x^2}\,dx$;
   (b) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$.

9. Using differentiation, show that the following function is constant on $(0, +\infty)$:
$$h(x) = \int_0^x \frac{dt}{t^2+1} + \int_0^{\frac{1}{x}} \frac{dt}{t^2+1}.$$

10. Check if the Mean Value Theorem holds for the following function on $[0, 2]$:
$$f(x) = \begin{cases} 0, & x < 1, \\ 1, & x \geq 1. \end{cases}$$
    Explain why.