Supplementary problem sheet 2

Differential calculus

1. Show that
   (a) the derivative of an odd function is even;
   (b) the derivative of an even function is odd;
   (c) the derivative of a periodic function is periodic.
   What can you say about the extrema and the inflection points of odd/even/periodic functions?
2. Consider the function
   $$\cos(x^2) \text{ on } [-2, 2].$$
   (a) Determine its points of local and global extremum.
   (b) How many inflection points does it have?
3. Give an example of a function which is differentiable once but not twice on a given interval.
4. Suppose that the functions f, g, and h are all differentiable at the point x_0. Express $(fgh)'(x_0)$ in terms of the values of the three functions and their derivatives at x_0.
5. Prove the formula for the derivative of the power function x^r, $r \in \mathbb{R}$, using logarithmic differentiation.
6. Show that for all real x one has
   $$\cos(x) \geq 1 - \frac{x^2}{2}.$$
   Hint: Show that it is sufficient to check that $f(x) \geq 0$ for a certain function f, and then determine the global minima of this f.
7. How many solutions does the equation $\cos(x) = x^2$ have?
8. Suppose that the function f satisfies the following conditions:
   (i) $f(x + y) = f(x)f(y)$ for all real x and y;
   (ii) f is differentiable at 0, and $f'(0) = 1$.
   Show that
   (a) f is differentiable everywhere, and $f'(x) = f(x)$ for all x;
   (b) when x varies from $-\infty$ to $+\infty$, $f(x)$ increases from 0 to $+\infty$.
   Remark. In fact, this exercise shows that the property "transforming sums into products" essentially determines the class of exponential functions.
9. Compute the derivative
   $$\left(\sqrt{1-x^2}\right)'$$
   in two ways:
   (a) using the rules of differential calculus;
   (b) interpreting it as the slope of the tangent line to the unit circle at the point $(x, \sqrt{1-x^2})$.
10. Analyse the function
    $$f(x) = e^{-\frac{x^2}{2}}$$
    following the algorithm you were given for rational functions. Sketch its graph.
    Remark. This function is called *Gaussian function*, or *bell function* (because of its shape). It is omnipresent in sciences: you can encounter it in statistics, signal and image processing, artificial neural networks, optical and microwave systems, and many more.

11. Find a rational function whose graph has a bell-like shape similar to the one you sketched in the previous question.

12. (a) Write down the recursive formula of Newton's method for the function

$$f(x) = \frac{1}{x} - a.$$

Here $a \neq 0$ is a fixed real number.

(b) Briefly explain how it can be used to approximate $\frac{1}{a}$ with the help of a device that can only add and multiply real numbers.

(c) Using this, approximate $\frac{1}{13}$.