Supplementary problem sheet 1

Functions: elementary properties, limits, continuity, derivatives

1. Consider the function $f(x) = \arcsin(\sin(x))$.
   (a) What is its natural domain?
   (b) Is f even? odd? periodic?
   (c) Is f continuous?
   (d) At what points is f differentiable? Compute its derivative f'.
   (e) Plot the graphs of f and f'.

2. Compute $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$, $\lim\limits_{x\to 0} \dfrac{1-\cos(x^2)}{x^4}$, and $\lim\limits_{x\to 0} \dfrac{\sin(2x^4)}{1-\cos(3x^2)}$.

   (*Hint.* Read the supplementary materials on the remarkable limit $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$, available on the web pages of the module.)

3. Compute $\lim\limits_{x\to +\infty} \sqrt[3]{x^2-5x} - \sqrt[3]{x^2+7x}$.

4. Construct a rational function f which satisfies all of the following properties:
   (a) its graph has two vertical asymptotes: $x = 0$ and $x = 2$;
   (b) its graph has a removable discontinuity at $x = -5$;
   (c) f is continuous for $x \neq -5, 0, 2$;
   (d) $\lim\limits_{x\to 0} f(x) = -\infty$;
   (e) its graph has the same horizontal asymptote at $+\infty$ and at $-\infty$: $y = 2$;
   (f) f has only one zero.
   Sketch the graph of f.

5. (a) Prove that if the function $f(x)$ is continuous everywhere, then so is the function $g(x) = |f(x)|$.
   (b) Give an example of a function $f(x)$ which is discontinuous at some points, but for which the function $g(x) = |f(x)|$ is continuous everywhere.

6. Consider the function $f(x) = x \sin \frac{1}{x}$.
   (a) Compute $\lim\limits_{x\to 0} f(x)$.
   (b) What is the discontinuity type of f at 0?
   (c) Is f differentiable at 0?
   (d) Does f have a limit at $\pm\infty$?

7. Consider a function $g(x)$ defined everywhere. Prove that $\lim\limits_{x\to 0}(\tan(x)\arctan(g(x))) = 0$.

8. Suppose that a function $f(x)$ is continuous on $(0, 1)$, and that $f(\frac{1}{2k}) = 3$ and $f(\frac{1}{2k+1}) = -1$ for all positive integers k. Show that f has zeroes as close to $x = 0$ as you wish. Give an example of such function. (By a **zero** of a function f we mean a value x for which $f(x) = 0$.)

9. Prove that at every moment one can find two opposite points on the Arctic Circle with the same air temperature.