Block Preconditioners for Incompressible Magnetohydrodynamics

Michael Wathen
STFC
Rutherford Appleton Laboratory

Beyond the discrete: iterative methods from the continuum perspective
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Joint work with Chen Greif, UBC
Outline

Linear systems

MHD background

Block Preconditioner

Approximate Inverse
Outline

Linear systems

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Approximate Inverse
Ax = b
Solution methods: direct or iterative?

Consider

\[ Ax = b \]

- We consider large and sparse linear systems
- Thus direct methods are impractical (based on factorizations) so an iterative approach is needed (based mat-vecs)
- Krylov subspace methods are state-of-the-art iterative techniques:

\[ x_k - x_0 \in \text{span}\{r_0, Ar_0, A^2r_0, \cdots, A^{k-1}r_0\} \]

where \( r_0 = b - Ax_0 \) is the initial residual
Preconditioning

Left-preconditioned linear system:

$$P^{-1}Ax = P^{-1}b$$

A preconditioner $P$ aims to speed up convergence of the iterative method

- Do this by eigenvalue clustering
- For discretized PDEs, aim for a constant number of iterations with respect to mesh (scalability)
Saddle point systems

Consider:

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]
Saddle point systems

Consider

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]

Common approaches are based on Schur complement preconditioners of the form:

\[
P_1 = \begin{pmatrix}
A & 0 \\
0 & BA^{-1}B^T
\end{pmatrix}
\text{ and } 
P_2 = \begin{pmatrix}
A + B^T W^{-1}B & 0 \\
0 & W
\end{pmatrix}
\]
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Approximate Inverse
MHD Model

- MHD models electrically conductive fluids (such as liquid metals, plasma, salt water, etc) in an electromagnetic field
- Applications: electromagnetic pumping, aluminum electrolysis, the Earth’s molten core and solar flares
- MHD couples electromagnetism (governed by Maxwell’s equations) and fluid dynamics (governed by the Navier-Stokes equations)
  - Motion of the conductive fluid induces and modifies the existing electromagnetic field
  - Electromagnetic field generates a force on the fluid
Continuous MHD model

Elliptic PDE in steady state

\[-\nu \Delta u + (u \cdot \nabla)u + \nabla p - \kappa (\nabla \times b) \times b = f \quad \text{in } \Omega,\]

\[\nabla \cdot u = 0 \quad \text{in } \Omega,\]

\[\kappa \nu_m \nabla \times (\nabla \times b) + \nabla r - \kappa \nabla \times (u \times b) = g \quad \text{in } \Omega,\]

\[\nabla \cdot b = 0 \quad \text{in } \Omega,\]

with appropriate boundary conditions.

\[\nabla \times (\nabla \times b) \times b: \text{ Lorentz force accelerates the fluid particles in the direction normal to the electric and magnetic fields}\]

\[\nabla \times (u \times b): \text{ electromotive force modifying the magnetic field}\]
Discretization: Finite Element Method

Weak formulation: Integrate \((u, p, b, r)\) against a set of test functions

\[
\int_{\Omega} \nu \nabla u \cdot \nabla v + \int_{\Omega} (u \cdot \nabla) u \cdot v + \int_{\Omega} \kappa (v \times b) \cdot \nabla \times b - \int_{\Omega} \nabla \cdot v \cdot p = \int_{\Omega} f \cdot v, \\
- \int_{\Omega} \nabla \cdot u \cdot q = 0, \\
\int_{\Omega} \kappa \nu m \nabla \times b \cdot \nabla \times c - \int_{\Omega} \kappa (u \times b) \cdot \nabla \times b + \int_{\Omega} c \cdot \nabla r = \int_{\Omega} g \cdot c, \\
\int_{\Omega} b \cdot \nabla s = 0.
\]
Discretization: Finite Element Method

Mixed finite elements: $H^1(\Omega) \times L^2(\Omega) \times H(\text{curl}, \Omega) \times H^1(\Omega)$
Discretized and Linearized MHD Model:

\[
\begin{pmatrix}
F & B^T & C^T & 0 \\
B & 0 & 0 & 0 \\
-C & 0 & M & D^T \\
0 & 0 & D & 0
\end{pmatrix}
\begin{pmatrix}
\delta u \\
\delta p \\
\delta b \\
\delta r
\end{pmatrix} =
\begin{pmatrix}
r_u \\
r_p \\
r_b \\
r_r
\end{pmatrix},
\]

\(C\): coupling terms; \(F\): convection–diffusion term; \(B\): fluid divergence operator; \(M\): curl-curl operator; \(D\): magnetic divergence operator
Numerical software

- **FEniCS**: finite element discretization (Sweden/USA/UK)
  - *mshr*: mesh generator (utilizing Tetgen and CGAL)

- **PETSc**: linear algebra backend (Argonne National Lab)
  - *Hypre*: AMG solver (Lawrence Livermore National Lab)
  - *MUMPS*: sparse direct solver (France)
MHD Preconditioners

- Adler, Benson, Cyr, MacLachlan, Tuminaro (2016) developed an “all-at-once” type multigrid solver based on Vanka smoothers for the 4-by-4 formulation.
- W., Greif (2019): approximate inverse-based preconditioner.
- Additional references can be found in my thesis.
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Block Preconditioner

Approximate Inverse
Ideal preconditioning

Non-singular (1, 1) block (as in Navier-Stokes)

\[ \mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} ; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & -BF^{-1}B^T \end{pmatrix} \]

Murphy, Golub & Wathen (2000) showed \( \mathcal{K}^{-1}\mathcal{K} \) has exactly one eigenvalues: 1
Ideal preconditioning

Non-singular \((1, 1)\) block (as in Navier-Stokes)

\[
\mathcal{K} = \begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} F & B^T \\ 0 & -BF^{-1}B^T \end{pmatrix}
\]

Murphy, Golub & Wathen (2000) showed \(\mathcal{K}^{-1}\mathcal{K}\) has exactly one eigenvalues: 1

\(M\) singular with nullity \(m\) (as in time-harmonic Maxwell)

\[
\mathcal{K} = \begin{pmatrix} M & D^T \\ D & 0 \end{pmatrix}; \quad \mathcal{P} = \begin{pmatrix} M + D^TW^{-1}D & 0 \\ 0 & 0 \end{pmatrix}
\]

Greif & Schötzau (2006) showed \(\mathcal{K}^{-1}\mathcal{K}\) has exactly two eigenvalues: \(\pm 1\)
Block preconditioning

Combine established preconditioners for the sub-problems

\[ M_{i}^{\text{MHD}} = \begin{pmatrix}
F & B^T & C^T & 0 \\
0 & -BF^{-1}B^T & 0 & 0 \\
-C & 0 & M + D^T L^{-1} D & 0 \\
0 & 0 & 0 & L
\end{pmatrix} \]
Block preconditioning

Combine established preconditioners for the sub-problems

\[ M_{i}^{MHD} = \begin{pmatrix}
F & C^T & B^T & 0 \\
-C & M + D^T L^{-1} D & 0 & 0 \\
0 & 0 & -B F^{-1} B^T & 0 \\
0 & 0 & 0 & L
\end{pmatrix}. \]
Block preconditioning

Combine established preconditioners for the sub-problems

\[
\mathcal{M}_{I}^{\text{MHD}} = \begin{pmatrix}
F & C^T & B^T & 0 \\
-C & M + D^T L^{-1} D & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -BF^{-1} B^T & 0
\end{pmatrix}.
\]

Combining fluid and magnet field using Schur complement technique

\[
\mathcal{M}_{S}^{\text{MHD}} = \begin{pmatrix}
F + M_C & C^T & B^T & 0 \\
0 & M + D^T L^{-1} D & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -BF^{-1} B^T & 0
\end{pmatrix}
\]

where \( M_C = C^T(M + D^T L^{-1} D)^{-1} C \)
Spectral Structure: Clustering Effect

**Red**: imaginary part of eigenvalues

**Blue**: real part of eigenvalues

Eigenvalues of the preconditioned matrix $(\mathcal{M}_I^{\text{MHD}})^{-1} \mathcal{K}$

Eigenvalues of the preconditioned matrix $(\mathcal{M}_S^{\text{MHD}})^{-1} \mathcal{K}$
## Results

<table>
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<th>$t_{\text{NL}}$</th>
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**Table:** 3D smooth: Number of nonlinear iterations and number of iterations to solve the MHD system with $\kappa = 1$, $\nu = 1$ and $\nu_m = 10$
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Approximate Inverse
Approximate Inverse

- Estrin & Greif (2015): an inverse formula for saddle-point systems with a maximally rank-deficient leading block

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\]

with \( \text{rank}(A) = n - m \), \( \text{rank}(B) = m \), and \( \text{ker}(A) \cap \text{ker}(B) = \{0\} \).

- The mixed Maxwell formulation used falls into this class of saddle-point systems

- If the leading block has a maximal nullity then the inverse has a zero (2,2) block and the other blocks can be represented by the null-space of the leading block
Discretized and Linearized Equations

Back to the equations we solve in the nonlinear iteration (with a slight change of notation):

\[
\begin{pmatrix}
F(u) & B^T & C(b)^T & 0 \\
B & 0 & 0 & 0 \\
-C(b) & 0 & M & DT \\
0 & 0 & D & 0
\end{pmatrix}
\begin{pmatrix}
\delta u \\
\delta p \\
\delta b \\
\delta r
\end{pmatrix}
=
\begin{pmatrix}
r_u \\
r_p \\
r_b \\
r_r
\end{pmatrix}
\]  
\} \begin{array}{c}n_u \text{ rows} \\ m_u \text{ rows} \\ n_b \text{ rows} \\ m_b \text{ rows}\end{array}

Let

\[
\mathcal{K}x = \begin{pmatrix}
\mathcal{K}_{NS} & \mathcal{K}_{C}^T \\
-\mathcal{K}_{C} & \mathcal{K}_{M}
\end{pmatrix}
\begin{pmatrix}
x_v \\
x_b
\end{pmatrix}
= \begin{pmatrix}
f_u \\
f_b
\end{pmatrix}
\]

where \(\mathcal{K}_{NS}, \mathcal{K}_{C}\) and \(\mathcal{K}_{M}\) are the Navier-Stokes, coupling and Maxwell block matrices.
General Inverse of Saddle Point Matrix

\[ A = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \]

\( A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{m \times n} \)

If \( A \) is non-singular then from Benzi, Golub & Liesen (2005)

\[ A^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B^TS^{-1}BA^{-1} & -A^{-1}B^TS^{-1} \\ -S^{-1}BA^{-1} & S^{-1} \end{pmatrix} \]

Will be useful for the Navier-Stokes block
Maximal Nullity of Leading Block

Estrin & Greif (2015): If $A$ has nullity $m$ then the (2,2) block of the inverse is zero. The inverse formula is

$$A^{-1} = \begin{pmatrix} A_W^{-1}(I - D^T W^{-1} G^T) & GW^{-1} \\ W^{-1} G^T & 0 \end{pmatrix},$$

where $W$ is a (free) symmetric positive definite matrix,

$$A_W = A + B^T W^{-1} B \quad \text{and} \quad G = A_W^{-1} B^T.$$

This comes handy for the block Schur complement associated with the $4 \times 4$ block MHD matrix.
Block Schur Complement

Then

\[
S = K_M + K_C K_{NS}^{-1} K_C^T = \begin{pmatrix} M + CK_1 C^T & D^T \\ D & 0 \end{pmatrix}
\]

Remark

Note that the null \( C^T \) and \( M \) have the same null space (discrete gradients). Therefore,

\[
dim(\text{null}(M + CK_1 C^T)) = m_b,
\]

where \( m_b \) is the number of rows of the magnetic discrete divergence matrix \( D \)
Block Schur Complement

Then

\[ S = \mathcal{K}_M + \mathcal{K}_C \mathcal{K}_{NS}^{-1} \mathcal{K}_C^T = \begin{pmatrix} M + CK_1 C^T & D^T \\ D & 0 \end{pmatrix} \]

Using Estrin & Greif (2015):

\[ S^{-1} = \begin{pmatrix} M_F^{-1} (I - D^T W^{-1} G^T) & GW^{-1} \\ W^{-1} G^T & 0 \end{pmatrix}, \]

where \( W \) is a (free) symmetric positive definite matrix,

\[ M_F = M + D^T W^{-1} D + CK_1 C^T \quad \text{and} \quad G = M_F^{-1} D^T. \]
Sparse Block Approximation

Sparsify utilizing:
1. Small mesh-based block elements
2. Null-space properties
3. Approximate Schur complements

Note: Never explicitly form the dense blocks
Eigenvalue Distribution

Figure: Eigenvalues of preconditioned matrix $P_1^{-1}K$ where red are the imaginary and blue are the real parts of the eigenvalues.
3D Cavity Driven Flow

\[ u = (1, 0, 0) \text{ on } z = 1, \]
\[ u = (0, 0, 0) \text{ on } x = \pm 1, y = \pm 1, z = -1, \]
\[ n \times b = n \times b_N \text{ on } \partial \Omega, \]
\[ r = 0 \text{ on } \partial \Omega, \]

where \( b_N = (-1, 0, 0) \).
3D Cavity Driven Flow

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>DoFs</th>
<th>time(^A)</th>
<th>( it_{NL}^A )</th>
<th>( it_{O}^A )</th>
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<td>58.5</td>
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</table>

Table: 3D Cavity Driven using both the approximate inverse and block triangular preconditioner with parameters \( \kappa = 1e1, \nu = 1e-1, \nu_m = 1e-1 \) and \( \text{Ha} = \sqrt{1000} \)
Fichera Corner

Figure: Example Fichera corner domain for mesh level $\ell = 3$
Table: Fichera corner using the approximate inverse preconditioner
\( \kappa = 1e1, \nu = 1e-2, \nu_m = 1e-2 \) and \( \text{Ha} = \sqrt{1e5} \).

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>DoFs</th>
<th>time(^A)</th>
<th>( it_{\text{NL}}^A )</th>
<th>( it_{\text{O}}^A )</th>
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<td>29.2</td>
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<td>5,232,365</td>
<td>11593.47</td>
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Thank you!