Krylov subspace methods and preconditioning in infinite dimensional Hilbert spaces
Introduction and Part I

Zdeněk Strakoš
Charles University, Prague
Jindřich Nečas Center for Mathematical Modelling

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Motto of the talk

Mathematical modeling, related analysis, discretization, and computation have to deal with questions that go across several fields. Handling them in their complexity requires extensive and thorough collaboration.
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When we consider a problem that results from mathematical modeling of some real world phenomena, it is useful to include into considerations from the very beginning also the views on how at the end the (approximate) solution of the discretized finite dimensional counterpart is going to be computed numerically.
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When we consider a problem that results from mathematical modeling of some real world phenomena, it is useful to include into considerations from the very beginning also the views on how at the end the (approximate) solution of the discretized finite dimensional counterpart is going to be computed numerically.

Vice versa, in challenging numerical computations it is beneficial to consider the path from the mathematical model through its discretization to the finite dimensional algebraic problem. In particular, numerical computation should respect structure present in the mathematical model. Moreover, difficulties faced in numerical computations should be carefully examined in terms of their sources in order to find out whether or not they are avoidable.
Following many others, Liesen, S (2013), Málek, S (2015)

Formulation of the model, discretization and algebraic computation, including the evaluation of the error, stopping criteria for the algebraic solver (here PCG), adaptivity etc. are very closely related to each other.
Many thanks to many coauthors, in particular to

Tomáš Gergelits,
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Bjørn Fredrik Nielsen,
Jan Papež,
Ivana Pultarová.
Outline

1. CG in infinite dimensional Hilbert spaces
2. Condition number and spectral number
3. Operator preconditioning
4. Discretization
5. Decomposition into subspaces and preconditioning
6. Coercivity and boundedness constants
7. Conclusions
More details and references to many original works can be found in

- J. Liesen. and Z.S., *Krylov Subspace Methods, Principles and Analysis*. Oxford University Press (2013), Sections 5.1 - 5.7

Hilbert space $V$ with the inner product

$$(\cdot, \cdot)_V : V \times V \to \mathbb{R}, \quad \Vert \cdot \Vert_V,$$

dual space $V^\#$ of bounded linear functionals on $V$ with the duality pairing and the associated Riesz map

$$\langle \cdot, \cdot \rangle : V^\# \times V \to \mathbb{R}, \quad \tau : V^\# \to V \quad \text{such that} \quad (\tau f, v)_V := \langle f, v \rangle \quad \text{for all} \ v \in V.$$

Equation in the functional space $V^\#$

$$A u = b$$

with a linear, bounded, coercive, and self-adjoint operator

$$A : V \to V^\# , \quad a(u, v) := \langle Au, v \rangle ,$$

$$C_A := \sup_{v \in V, \Vert v \Vert_V = 1} \Vert Av \Vert_{V^\#} < \infty ,$$

$$c_A := \inf_{v \in V, \Vert v \Vert_V = 1} \langle Av, v \rangle > 0 .$$
Using the Riesz map, $\tau A : V \to V$. One can form for $g \in V$ the Krylov sequence

$$g, \tau Ag, (\tau A)^2g, \ldots \text{ in } V$$

and define Krylov subspace methods in the Hilbert space operator setting (here CG) such that with $r_0 = b - Ax_0 \in V^\#\text{ the approximations } x_n\text{ to the solution } x, \ n = 1, 2, \ldots \text{ belong to the Krylov subspaces in } V$

$$x_n \in x_0 + K_n(\tau A, \tau r_0) \equiv x_0 + \text{span}\{\tau r_0, \tau A(\tau r_0), (\tau A)^2(\tau r_0), \ldots, (\tau A)^{n-1}(\tau r_0)\}.$$
Using the Riesz map, \( \tau A : V \rightarrow V \). One can form for \( g \in V \) the Krylov sequence

\[
g, \tau Ag, (\tau A)^2g, \ldots \quad \text{in } V
\]

and define Krylov subspace methods in the Hilbert space operator setting (here CG) such that with \( r_0 = b - Ax_0 \in V^\# \) the approximations \( x_n \) to the solution \( x \), \( n = 1, 2, \ldots \) belong to the Krylov subspaces in \( V \)

\[
x_n \in x_0 + K_n(\tau A, \tau r_0) \equiv x_0 + \text{span}\{\tau r_0, \tau A(\tau r_0), (\tau A)^2(\tau r_0), \ldots, (\tau A)^{n-1}(\tau r_0)\}.
\]

Approximating the solution \( x = (\tau A)^{-1}\tau b \) using Krylov subspaces is different from approximating the operator inverse \( (\tau A)^{-1} \) by the operators \( I, \tau A, (\tau A)^2, \ldots \).
Using the Riesz map, $\tau A : V \to V$. One can form for $g \in V$ the Krylov sequence
\[ g, \tau A g, (\tau A)^2 g, \ldots \] in $V$

and define Krylov subspace methods in the Hilbert space operator setting (here CG) such that with $r_0 = b - Ax_0 \in V^\#$ the approximations $x_n$ to the solution $x$, $n = 1, 2, \ldots$ belong to the Krylov subspaces in $V$

\[ x_n \in x_0 + K_n(\tau A, \tau r_0) \equiv x_0 + \text{span}\{\tau r_0, \tau A(\tau r_0), (\tau A)^2(\tau r_0), \ldots, (\tau A)^{n-1}(\tau r_0)\} \]

Approximating the solution $x = (\tau A)^{-1}\tau b$ using Krylov subspaces is different from approximating the operator inverse $(\tau A)^{-1}$ by the operators $I, \tau A, (\tau A)^2, \ldots$

In $\mathbb{R}^N$ the identification of the space with its dual can lead to misplacement of the residual to $V$ instead of $V^\#$. 
Defining the energy functional

\[ J(v) := \frac{1}{2} \langle Av, v \rangle - \langle b, v \rangle, \quad v \in V, \]

the solution is equivalently given by the condition

\[ x \in V \text{ minimizes the functional } J \text{ over } V. \]

The Galerkin solution (of the discretized problem) then solves

\[ x_h \in V_h \text{ minimizes the functional } J \text{ over } V_h. \]
Minimization of the energy functional

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There is a flexibility in considering \( V_h \). Minimization of the energy functional over the sequence of Krylov subspaces defines the iterates of the conjugate gradient method in infinite dimensional Hilbert space.
Using \( \|z\|^2_a = a(z, z) := \langle Az, z \rangle \), the approximate solution \( x_n \) minimizing the energy functional \( J \) over \( x_0 + K_n(\tau A, \tau r_0) \) is equivalently expressed as

\[
\|x - x_n\|_a = \min_{z \in x_0 + K_n} \|x - z\|_a,
\]

or by the (Galerkin) orthogonality condition

\[
\langle b - Ax_n, w \rangle = \langle r_n, w \rangle = 0 \quad \text{for all } w \in K_n(\tau A, \tau r_0).
\]

Since \( K_n(\tau A, \tau r_0) \) is finite dimensional, this provides in a straightforward way the discretization of the problem matching the maximal number of \( 2n \) moments, see below.
Starting from

\[ p_0 = \tau r_0 \in V, \]

we will construct a sequence of direction vectors \( p_0, p_1, \ldots \) and a sequence of scalars \( \alpha_0, \alpha_1, \ldots \) such that

\[ x_n = x_{n-1} + \alpha_{n-1} p_{n-1}, \quad n = 1, 2, \ldots \]

Here \( \alpha_{n-1} \) ensures the minimization of \( F(z) \) along the line \( z(\alpha) = x_{n-1} + \alpha p_{n-1} \).
If, in addition,

\[ K_n(\tau A, \tau r_0) = \text{span}\{p_0, p_1, \ldots, p_{n-1}\} \quad \text{and} \quad p_i \perp p_j, \ i \neq j, \]

then the one-dimensional line minimizations along the given \( n \) individual lines will be equivalent to minimization of the functional \( J \) over the whole \( n \) dimensional subspace \( x_0 + K_n(\tau A, \tau r_0) \), and

\[
x - x_0 = \sum_{\ell=0}^{n-1} \alpha_\ell p_\ell + (x - x_n)
\]

represents the orthogonal expansion of the initial error \( x - x_0 \) along the direction vectors \( p_0, \ldots, p_{n-1} \) with

\[
(x - x_n) \perp_a K_n.
\]
Minimizing $J$: the steepest descent direction $d_\ell \in V$ maximizes size of the directional derivative,

$$
\delta J(x_\ell; d_\ell) = \lim_{\nu \to 0} \frac{J(x_\ell + \nu d_\ell) - J(x_\ell)}{\nu} = -\langle r_\ell, d_\ell \rangle,
$$

$r_\ell = b - Ax_\ell \in V^\#$, while $d_\ell \in V$. It depends on the inner product $(\cdot, \cdot)_V$,

$$
-\langle r_\ell, d_\ell \rangle = - (\tau r_\ell, d_\ell)_V \quad \text{and} \quad d_\ell = \tau r_\ell.
$$

Then

$$p_n = \tau r_n + \beta_n p_{n-1}$$

with $\beta_n$ determined by the condition $p_n \perp_a p_{n-1}$. 

1 Search vectors (Hackbusch (1994)), scaling is unimportant
For $n = 1, 2, \ldots, n_{\text{max}}$, \quad (r_0 = b - Ax_0 \in V^\#, \quad p_0 = \tau r_0 \in V)$

\[
\alpha_{n-1} = \frac{\langle r_{n-1}, \tau r_{n-1} \rangle}{\langle Ap_{n-1}, p_{n-1} \rangle} = \frac{(\tau r_{n-1}, \tau r_{n-1})_V}{(\tau Ap_{n-1}, p_{n-1})_V}
\]

\[
x_n = x_{n-1} + \alpha_{n-1} p_{n-1}, \quad \text{stop when the stopping criterion is satisfied}
\]

\[
r_n = r_{n-1} - \alpha_{n-1} Ap_{n-1}
\]

\[
\beta_n = \frac{\langle r_n, \tau r_n \rangle}{\langle r_{n-1}, \tau r_{n-1} \rangle} = \frac{(\tau r_n, \tau r_n)_V}{(\tau r_{n-1}, \tau r_{n-1})_V}
\]

\[
p_n = \tau r_n + \beta_n p_{n-1}
\]

End

Karush (1952); Hayes (1954); Stesin (1954); Daniel (1967, 1967); ... , Fortuna (1979); Ernst (2000); Axelsson and Karatson (2002); Glowinski (2003); .... ; Zulehner (2011); Günnel, Herzog, and Sachs (2012); ... ; Vorobyev (1958, 1965)
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Preconditioning of a linear algebraic system

\[ A \mathbf{x} = \mathbf{b} \]

means its transformation to another system with more favourable properties for its iterative numerical solution. Standard textbook introduction considers \( A \) SPD and takes an SPD matrix \( B \approx A \) with decomposition \( B = LL^* \), giving

\[ L^{-1} AL^{-1} L^* \mathbf{x} = L^{-1} \mathbf{b} \, . \]

In order to apply an iterative method (CG) to the transformed system, its algorithm is reformulated in terms of the original variables (see preconditioned CG), which is better resembled by

\[ B^{-1} A \mathbf{x} = B^{-1} \mathbf{b} \, . \]
For later convenience, consider the special (reference) choice

\[ B = B^{1/2} B^{1/2}. \]

Then for any other decomposition \( B = L L^* \) we have

\[ L^{-1} B L^{-1} = (L^{-1} B^{1/2})(B^{1/2} L^{-1}) = I, \]

and taking the unitary matrix

\[ Q := L^{-1} B^{1/2}, \quad Q^{-1} = Q^* \rightarrow B^{-1/2} L = B^{1/2} L^{-1}, \]

we have the unitary transformation from \( L \) to \( B^{1/2} \) and vice versa

\[ L = B^{1/2} Q^*, \quad B^{1/2} = L Q. \]
For SPD matrix $A$

$$\kappa(A) := \|A\| \|A^{-1}\| = \frac{\sigma_{max}(A)}{\sigma_{min}(A)} = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}.$$
2 Matrices as operators

For SPD matrix $A$

$$\kappa(A) := \|A\|\|A^{-1}\| = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

Analogously for the preconditioned operator $L^{-1}AL^*^{-1}$, but not for the nonsymmetric preconditioned operator $B^{-1}A$

$$\kappa(B^{-1}A) := \|B^{-1}A\|\|A^{-1}B\| = \frac{\sigma_{\max}(B^{-1}A)}{\sigma_{\min}(B^{-1}A)} \neq \frac{\lambda_{\max}(B^{-1}A)}{\lambda_{\min}(B^{-1}A)} =: \hat{\kappa}(A,B).$$

We call $\hat{\kappa}(A,B)$ the spectral number of the pair $A,B$. Obviously

$$\hat{\kappa}(A,B) = \kappa(L^{-1}AL^*^{-1}).$$
2 Adaptation as the main principle of modern iterative methods

Cornelius Lanczos, March 9, 1947

“To obtain a solution in very few steps means nearly always that one has found a way that does justice to the inner nature of the problem.”

Albert Einstein, March 18, 1947

“Your remark on the importance of adapted approximation methods makes very good sense to me, and I am convinced that this is a fruitful mathematical aspect, and not just a utilitarian one.”

- Inner nature of the problem?
- Nonlinear adaptation of the iterations to linear problems!
- No single number characteristic can capture nonlinear adaptation.
Consider $2n$ positive numbers $m_0, m_1, \ldots, m_{2n-1}$.

Determine under which conditions the solution of the system of $2n$ equations

$$\sum_{j=1}^{n} \omega_j^{(n)} \{\theta_j^{(n)}\}^\ell = m_\ell, \quad \ell = 0, 1, \ldots, 2n - 1,$$

for the $2n$ positive unknowns $\omega_j^{(n)} > 0$, $\theta_j^{(n)} > 0$, $j = 1, \ldots, n$ exists and is unique, and give the solution.
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$$

for the $2n$ positive unknowns $\omega_j^{(n)} > 0$, $\theta_j^{(n)} > 0$, $j = 1, \ldots, n$ exists and is unique, and give the solution.

Is this problem linear?
Mathematical description of the solution?
How to compute the solution?
Consider an infinite sequence of positive numbers $m_0, m_1, m_2, \ldots$.

Find necessary and sufficient conditions for the existence of a Riemann-Stieltjes integral with the (positive nondecreasing) distribution function $\omega(\lambda)$ such that

$$\int_{0}^{\infty} \lambda^\ell \, d\omega(\lambda) = m_\ell, \quad \ell = 0, 1, 2, \ldots$$

and determine the distribution function $\omega(\lambda)$.

Let $G : V \to V$ be a linear bounded self-adjoint operator on a Hilbert space $V$, $f \in V$, $\|f\| = 1$. Consider the $2n$ real numbers

$$m_j = (G^j f, f) \quad \left( = \int \lambda^j \, d\omega(\lambda) \right), \quad j = 0, \ldots, 2n - 1.$$
Let $\mathcal{G} : V \to V$ be a linear bounded self-adjoint operator on a Hilbert space $V$, $f \in V$, $\|f\| = 1$. Consider the $2n$ real numbers

$$m_j = (\mathcal{G}^j f, f) = \int \lambda^j d\omega(\lambda), \quad j = 0, \ldots, 2n - 1.$$

The first $n$ steps of CG (assuming, in addition, the coercivity of $\mathcal{G}$), as well as of the Lanczos method for approximating eigenvalues, determine (implicitly) the solution $\omega_j^{(n)} > 0, \theta_j^{(n)} > 0$, $j = 1, \ldots, n$ of the $2n$ equations

$$\sum_{j=1}^n \omega_j^{(n)} \{\theta_j^{(n)}\}_\ell = m_\ell, \quad \ell = 0, 1, \ldots, 2n - 1.$$

- Golub, Welsch (1968), Gordon (1968), ... , Vorobyev (1958, 1965)
- An overview with generalization to complex Gauss quadrature is given in Pozza, Pranic, S (2016), an overview of the relationship with the nonsymmetric Lanczos algorithm in Pozza, Pranic, S (2017); many results in the works of Stieltjes, ... , Draux, Gragg, Kautsky, Gutknecht, Brezinski, ...
It is very useful to investigate condition and spectral numbers in the context of preconditioning and iterative methods, providing that we do not consider such investigations the end of the story.
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1. CG in infinite dimensional Hilbert spaces
2. Condition number and spectral number
3. **Operator preconditioning**
4. Discretization
5. Decomposition into subspaces and preconditioning
6. Coercivity and boundedness constants
7. Conclusions
Gunn, D’yakonov, Faber, Manteuffel, Parter, Klawonn, Arnold, Falk, Winther, Axelsson, Karátson, Hiptmair, Vassilevski, Neytcheva, Notay, Elmann, Silvester, Wathen, Zulehner, Simoncini, Oswald, Griebel, Rüde, Steinbach, Wohlmuth, Bramble, Pasciak, Xu, Kraus, Nepomnyaschikh, Dahmen, Kunoth, Yserentant, Mardal, Nordbotten, Herzog, Sachs, Pestana, Pearson, Powell, Smears, .........

Details, proofs and (certainly incomplete) references can be found in

3 Recall basic setting

Hilbert space $V$ with the inner product

$$(\cdot, \cdot)_V : V \times V \to \mathbb{R}, \quad \| \cdot \|_V,$$

dual space $V^\#$ of bounded linear functionals on $V$ with the duality pairing and the associated Riesz map

$$\langle \cdot, \cdot \rangle : V^\# \times V \to \mathbb{R}, \quad \tau : V^\# \to V \quad \text{such that} \quad (\tau f, v)_V := \langle f, v \rangle \quad \text{for all} \ v \in V.$$ 

Equation in the functional space $V^\#$

$$A u = b$$

with a linear, bounded, coercive, and self-adjoint operator

$$A : V \to V^\# , \quad a(u, v) := \langle Au, v \rangle ,$$

$$C_A := \sup_{v \in V, \| v \|_V = 1} \| Au \|_{V^\#} < \infty ,$$

$$c_A := \inf_{v \in V, \| v \|_V = 1} \langle Av, v \rangle > 0 .$$
Linear, bounded, coercive, and self-adjoint $B, C_B, c_B$ defined analogously. Define

$$(\cdot, \cdot)_B : V \times V \to \mathbb{R}, \quad (w, v)_B := \langle Bw, v \rangle \quad \text{for all } w, v \in V,$$

$$\tau_B : V^\# \to V, \quad (\tau_B f, v)_B := \langle f, v \rangle \quad \text{for all } f \in V^\#, \ v \in V.$$ 

Instead of the equation in the functional space $V^\#

$$Au = b$$

we solve the equation in the solution space $V$

$$\tau_B Au = \tau_B b,$$

i.e.

$$B^{-1} Au = B^{-1} b.$$
Theorem 3.1 (Norm equivalence and condition number)

Assuming that the linear, bounded, coercive and self-adjoint operators $A$ and $B$ are $V^\#$-norm equivalent on $V$, i.e. there exist $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq \frac{\|Aw\|_{V^\#}}{\|Bw\|_{V^\#}} \leq \beta, \quad \text{for all } w \in V, w \neq 0.$$

Then

$$\kappa(B^{-1}A) := \|B^{-1}A\|_{\mathcal{L}(V,V)} \|A^{-1}B\|_{\mathcal{L}(V,V)} \leq \frac{\beta}{\alpha}.$$ 

cf. Faber, Manteuffel, Parter (1990)
Theorem 3.3 (Spectral equivalence and spectral number)

Assuming that the linear, bounded, coercive and self-adjoint operators $A$ and $B$ are spectrally equivalent on $V$, i.e. there exist $0 < \gamma \leq \delta < \infty$ such that

$$\gamma \leq \frac{\langle Aw, w \rangle}{\langle Bw, w \rangle} \leq \delta, \quad \text{for all } w \in V, w \neq 0.$$  

Then

$$\hat{\kappa}(A, B) := \frac{\sup_{z \in V, \|z\|_V = 1} \left( (\tau B)^{-1/2} \tau A (\tau B)^{-1/2} z, z \right)_V}{\inf_{v \in V, \|v\|_V = 1} \left( (\tau B)^{-1/2} \tau A (\tau B)^{-1/2} v, v \right)_V} \leq \frac{\delta}{\gamma}.$$
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Consider $N$-dimensional subspace $V_h \subset V$; and look for $u_h \in V_h$, $u_h \approx u \in V$ such that
\[
\langle Au_h - b, v \rangle = 0 \quad \text{for all } v \in V_h.
\]

Restrictions $A_h : V_h \to V_h^\#$, $b_h : V_h \to \mathbb{R}$ give the problem in $V_h^\#
\[
A_h u_h = b_h, \quad u_h \in V_h, \quad b_h \in V_h^\#.
\]

With the inner product $(\cdot, \cdot)_B$ and the associated restricted Riesz map
\[
\tau_{B,h} : V_h^\# \to V_h
\]
we get the abstract form of the preconditioned discretized problem in $V_h$
\[
\tau_{B,h} A_h u_h = \tau_{B,h} b_h.
\]
Using the discretization basis $\Phi_h = (\phi_1, \ldots, \phi_N)$ of $V_h$ and the canonical dual basis $\Phi^\# = (\phi_1^\#, \ldots, \phi_N^\#)$ of $V^\#$, $(\Phi_h^\#)^* \Phi_h = I_N$, we have

$$M_h^{-1} A_h x_h = M_h^{-1} b_h,$$

where

$$A_h, M_h \in \mathbb{R}^{N \times N}, \quad x_h, b_h \in \mathbb{R}^N,$$

$$(x_h)_i = \langle \phi_i^\#, u_h \rangle, \quad (b_h)_i = \langle b, \phi_i \rangle,$$

$$A_h = (\langle A \phi_j, \phi_i \rangle)_{i,j=1,\ldots,N},$$

$$M_h = (\langle B \phi_j, \phi_i \rangle)_{i,j=1,\ldots,N},$$

or

$$A_h = (A \Phi_h)^* \Phi_h, \quad M_h = (B \Phi_h)^* \Phi_h.$$
Indeed, for the restricted Riesz map $\tau_{B,h}$, $v$ and $f$ with $f = \Phi_h^# f$, $v = \Phi_h v$,

\[(\tau_{B,h} f, v)_B = (\tau_{B,h} \Phi_h^# f, \Phi_h v)_B = (\Phi_h M_{\tau} f, \Phi_h v)_B = \langle B \Phi_h M_{\tau} f, \Phi_h v \rangle = v^* M_h M_{\tau} f,\]

\[(\tau_{B,h} f, v)_B = \langle f, v \rangle = v^* f\]

and therefore

\[M_{\tau} = M_h^{-1}.\]

Using (an arbitrary) decomposition $M_h = L_h L_h^*$, the resulting preconditioned algebraic system can be transformed into

\[(L_h^{-1} A_h L_h^{-1}) (L_h^* x_h) = L_h^{-1} b_h,\]

i.e.,

\[A_{t,h} x_{h}^t = b_h^t.\]
Consider orthogonalization of the basis with respect to the inner product \((\cdot, \cdot)_B\),

\[ \Phi_h \rightarrow \tilde{\Phi}_{t,h} \quad \text{such that} \quad M_{t,h} = (B\tilde{\Phi}_{t,h})^*\tilde{\Phi}_{t,h} = I. \]

Then the choice of the bases

\[ \tilde{\Phi}_{t,h} = \Phi_h M_h^{-1/2}, \quad \tilde{\Phi}_#_{t,h} = \Phi_h M_h^{1/2} \]

results in the preconditioned system \( \tilde{A}_{t,h} \tilde{x}_h^t = \tilde{b}_h^t \) corresponding to \( L_h := M_h^{1/2} \). Any other choice \( M_h =: L_h L_h^* \) gives

\[ \Phi_{t,h} = \Phi_h L_h^{*-1}, \quad \Phi^#_{t,h} = \Phi_h^# L_h, \]

which is linked via the orthogonal transformation

\[ \Phi_{t,h} = \tilde{\Phi}_{t,h} Q^*, \quad Q^* = M_h^{1/2} L_h^{*-1}, \quad Q^* Q = I. \]
Preconditioning is mathematically equivalent to orthogonalization of the discretization basis with respect to the inner product \((\cdot, \cdot)_B\). This will change the supports of the basis functions!

Transformation of the discretization basis (preconditioning) is different from a change of the algebraic basis (similarity transformation).

Any algebraic preconditioning can be put into the operator preconditioning framework by transformation of the discretization basis and the associated change of the inner product in the infinite dimensional Hilbert space \(V\).
Consider an algebraic preconditioning with the (SPD) preconditioner

\[
\hat{M} = \hat{L}\hat{L}^* = \hat{L} (QQ^*) \hat{L}^*
\]

Where \( QQ^* = Q^* Q = I \).
Transform the discretization bases

$$\hat{\Phi} = \Phi ((\hat{LQ})^*)^{-1}, \quad \hat{\Phi}^# = \Phi^# \hat{LQ}. $$

with the simultaneous change of the inner product in $V$ (recall $(u, v)_V = v^*Mu$)

$$(u, v)_{\text{new}, V} = (\hat{\Phi}\hat{u}, \hat{\Phi}\hat{v})_{\text{new}, V} := \hat{v}^*\hat{u} = v^*\hat{LQQ}^*\hat{L}^*u = v^*\hat{L}\hat{L}^*u = v^*\hat{M}u. $$
Transform the discretization bases

\[ \hat{\Phi} = \Phi ((\hat{L}Q)^*)^{-1}, \quad \hat{\Phi}# = \Phi# \hat{L}Q. \]

with the simultaneous change of the inner product in \( V \) (recall \( (u, v)_V = v^*Mu \))

\[ (u, v)_{\text{new}, V} = (\hat{\Phi}u, \hat{\Phi}v)_{\text{new}, V} := \hat{v}^*\hat{u} = v^*\hat{L}QQ^*\hat{L}^*u = v^*\hat{L}\hat{L}^*u = v^*\hat{M}u. \]

Then the discretized Hilbert space formulation of CG results in the algebraically preconditioned matrix formulation of CG with the preconditioner \( \hat{M} \) (more specifically, it results in the unpreconditioned CG applied to the algebraically preconditioned discretized system).
Sparsity of matrices of the algebraic systems is always presented as an advantage of the FEM discretizations.
Sparsity of matrices of the algebraic systems is always presented as an advantage of the FEM discretizations.

Sparsity means locality of information in the individual matrix rows/columns. Getting a sufficiently accurate approximation to the solution may then require many matrix-vector multiplications (a large dimension of the Krylov subspace).
Sparsity of matrices of the algebraic systems is always presented as an advantage of the FEM discretizations.

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Globally supported basis functions (hierarchical bases preconditioning, DD with coarse space components, multilevel methods, hierarchical grids etc.) can efficiently handle the transfer of global information.
4 Transformed (linear) FEM nodal basis elements have global support

Transformed discretization basis elements corresponding to the \texttt{lapl} (left) and \texttt{ichol(tol)} preconditioning (right).
Theorem 4.1 (Norm equivalence and condition number)

Let the linear, bounded, coercive and self-adjoint operators $A$ and $B$ from $V$ to $V^\#$ be $V^\#$-norm equivalent with the lower and upper bounds $\alpha$ and $\beta$, respectively, i.e.

$$\alpha \leq \frac{\|Aw\|_{V^\#}}{\|Bw\|_{V^\#}} \leq \beta \quad \text{for all } w \in V, \ w \neq 0, \quad 0 < \alpha \leq \beta < \infty.$$ 

Let $S_h$ be the Gram matrix of the discretization basis $\Phi_h = (\phi_1, \ldots, \phi_N)$ of $V_h \subset V$, with $(\Phi_h^\#)^* \Phi_h = I$,

$$(S_h)_{ij} = (\phi_i, \phi_j)_V.$$ 

Then the condition number of the matrix $M_h^{-1}A_h$ is bounded as

$$\kappa(M_h^{-1}A_h) := \|M_h^{-1}A_h\| \|A_h^{-1}M_h\| \leq \frac{\beta}{\alpha} \kappa(S_h).$$
Theorem 4.2 (Spectral equivalence and spectral number)

Let the linear, bounded, coercive and self-adjoint operators $A$ and $B$ be spectrally equivalent with the lower and upper bounds $\gamma$ and $\delta$ respectively, i.e.

$$\gamma \leq \frac{\langle Aw, w \rangle}{\langle Bw, w \rangle} \leq \delta \quad \text{for all } w \in V, \quad 0 < \gamma \leq \delta < \infty.$$ 

Then the spectral number $\hat{\kappa}(A_h, M_h)$, which is equal to the condition number of the matrix $A_{t,h} = L_h^{-1} A_h (L_h^*)^{-1}$ for any $L_h$ such that $M_h = L_h L_h^*$, is bounded as

$$\hat{\kappa}(A_h, M_h) := \sup_{z \in \mathbb{R}^N, \|z\| = 1} \left( \frac{M_h^{-1/2} A_h M_h^{-1/2} z, z}{\kappa(A_{t,h}) \leq \frac{\delta}{\gamma}} \right) = \kappa(A_{t,h}) \leq \frac{\delta}{\gamma}.$$
Here

$$\alpha, \beta, \delta, \gamma$$

are determined at the infinite dimensional level and they are not affected by discretization.
Outline

1. CG in infinite dimensional Hilbert spaces
2. Condition number and spectral number
3. Operator preconditioning
4. Discretization
5. Decomposition into subspaces and preconditioning
6. Coercivity and boundedness constants
7. Conclusions
Decomposition with non-unique representation of elements in \( V \)

\[
V = \sum_{j \in J} V_j, \quad \text{i.e.,} \quad v = \sum_{j \in J} v_j, \quad v_j \in V_j, \quad \text{for all} \ v \in V, \ J \text{ is finite;}
\]

Sufficient condition for \( V^\# \subset V_j^\# : \)

\[
cv_j \|v\|_V^2 \leq \|v\|_j^2 \quad \text{for all} \ v \in V_j, \ 0 < cv_j, \ j \in J;
\]

Other side inequality:

\[
\|v\|_S^2 := \inf_{v=\sum_{j \in J} v_j} \left\{ \sum_{j \in J} \|v_j\|_j^2 \right\} \leq C_S \|v\|_V^2, \quad \text{for all} \ v \in V.
\]
Consider local preconditioners

\[ B_j : V_j \to V_j^\# , \quad \langle B_j w, z \rangle = \langle B_j z, w \rangle \quad \text{for all } w, z \in V_j , \]

with \( C_{B_j} , c_{B_j} \) defined as above. Then \( B_j^{-1} : V_j^\# \to V_j , \quad V^\# \subset V_j^\# , \) and

\[ M^{-1} := \sum_{j \in J} B_j^{-1} , \quad M^{-1} : V^\# \to V \]

gives the global preconditioner. The preconditioned (equivalent?) problem

\[ M^{-1} A u = M^{-1} b . \]
5 Equivalence of the preconditioned system

Boundedness and coercivity of \( M^{-1} \)

\[
\|M^{-1}\|_{\mathcal{L}(V\#, V)} = \sup_{f \in V\#, \|f\|_{V\#} = 1} \|M^{-1}f\|_V \leq C_{M^{-1}} := \sum_{j \in J} \frac{1}{c_{B_j} c_{V_j}} < \infty,
\]

\[
\inf_{f \in V\#, \|f\|_{V\#} = 1} \langle f, M^{-1}f \rangle \geq c_{M^{-1}} := \frac{1}{C_S \max_{j \in J} C_{B_j}} > 0,
\]

gives equivalence of \( Au = b \) and \( M^{-1}Au = M^{-1}b \).

We can get norm equivalence and spectral equivalence of \( A \) and \( M \) and apply results presented above.
5 Bound using norms of the locally preconditioned residuals

**Theorem**

For any $v \in V \approx u$

$$a \left( \mathcal{M}^{-1} A (v - u), v - u \right) = \sum_{j \in J} \| \bar{r}_j \|_{B_j}^2,$$

$$\frac{\min_{j \in J} cB_j}{C_A^2} \left( \sum_{k \in J} \frac{1}{cV_k cB_k} \right)^{-1} \frac{\sum_{j \in J} \| \bar{r}_j \|_{B_j}^2}{\sum_{j \in J} \| \bar{r}_j \|_{B_j}^2} \leq \frac{C_S (\max_{j \in J} C_B) ^2}{c^2 A} \sum_{j \in J} \| \bar{r}_j \|_{B_j}^2,$$

where $\bar{r}_j := B_j^{-1} A v - B_j^{-1} b$ are the locally preconditioned residuals of $v$. 
5 Stable splitting

Theorem

If we consider the stable splitting

\[ c_S \|v\|_V^2 \leq \|v\|_S^2 \leq C_S \|v\|_V^2 \quad \text{for all } v \in V, \]

then

\[
\frac{c_A}{C_S \max_{j \in J} C_{B_j}} \leq \frac{\langle Av, v \rangle}{\langle Mv, v \rangle} \leq \frac{C_A}{c_S \min_{j \in J} c_{B_j}} \quad \text{for all } v \in V, v \neq 0,
\]

\[
\frac{c_S \min_{j \in J} c_{B_j}}{C_A} \leq \frac{\|A^{-1}f\|_V}{\|M^{-1}f\|_V} \leq \frac{C_S \max_{j \in J} C_{B_j}}{c_A} \quad \text{for all } f \in V^#, f \neq 0.
\]
- Domain decomposition with overlapping subdomains
- Separate displacement preconditioning in linear elasticity (Blaheta (1994))
- Additive algebraic multilevel preconditioning
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6 Coercivity and boundedness constants - consistent expressions

Theorem

Let $\mathcal{A}: V \to V^\#$ be a linear, bounded, coercive and self-adjoint operator. Then its boundedness constant $C_A$ and the coercivity constant $c_A$ can be expressed as

\[ C_A = \|\mathcal{A}\|_{\mathcal{L}(V,V^\#)} = \sup_{v \in V, \|v\|_V = 1} \langle \mathcal{A}v, v \rangle, \quad (1) \]

\[ c_A = \inf_{v \in V, \|v\|_V = 1} \langle \mathcal{A}v, v \rangle = \frac{1}{\sup_{f \in V^\#, \|f\|_{V^\#} = 1} \|\mathcal{A}^{-1}f\|_V} \]

\[ = \frac{1}{\|\mathcal{A}^{-1}\|_{\mathcal{L}(V^\#, V)}}. \quad (2) \]
Statement (1) follows from

\[ \|A\|_{\mathcal{L}(V,V^\#)} = \|\tau A\|_{\mathcal{L}(V,V)} = \sup_{v \in V, \|v\|_V = 1} (\tau Av, v)_V = \sup_{v \in V, \|v\|_V = 1} \langle Av, v \rangle, \]

where we used the fact that for any self-adjoint operator \( S \) in a Hilbert space \( V \)

\[ \|S\|_{\mathcal{L}(V,V)} = \sup_{z \in V, \|z\|_V = 1} \|Sz\|_V = \sup_{z \in V, \|z\|_V = 1} (Sz, Sz)_V^{1/2} = \sup_{z \in V, \|z\|_V = 1} |(Sz, z)_V|. \]
\[ \sup_{f \in \mathcal{V}^\#} \frac{1}{\|A^{-1} f\|_\mathcal{V}} = \inf_{v \in \mathcal{V}, \|v\|_\mathcal{V}=1} \|Av\|_{\mathcal{V}^\#} = \inf_{v \in \mathcal{V}, \|v\|_\mathcal{V}=1} \|\tau Av\|_\mathcal{V} \]

We have to prove

\[ m_A := \inf_{v \in \mathcal{V}, \|v\|_\mathcal{V}=1} (\tau Av, v)_\mathcal{V} = \inf_{v \in \mathcal{V}, \|v\|_\mathcal{V}=1} \|\tau Av\|_\mathcal{V}. \]

Here \( \leq \) is trivial. We will show that \( < \) leads to a contradiction. Since \( m_A \) belongs to the spectrum of \( \tau A \), there exists a sequence \( v_1, v_2, \ldots \in \mathcal{V}, \|v_k\|_\mathcal{V} = 1 \), such that

\[ \lim_{k \to \infty} \|\tau Av_k - m_A v_k\|_\mathcal{V}^2 = 0. \quad (3) \]

Assuming

\[ m_A < \inf_{v \in \mathcal{V}, \|v\|_\mathcal{V}=1} \|\tau Av\|_\mathcal{V} - \triangle, \quad \triangle > 0, \]

we get

\[ \|\tau Av_k - m_A v_k\|_\mathcal{V}^2 = \|\tau Av_k\|_\mathcal{V}^2 + m_A^2 - 2m_A(\tau Av_k, v_k)_\mathcal{V} \]
\[ \geq \|\tau Av_k\|_\mathcal{V}^2 + m_A^2 - 2m_A\|\tau Av_k\|_\mathcal{V} = (\|\tau Av_k\|_\mathcal{V} - m_A)^2 > \triangle^2. \]
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Operator preconditioning is the way to go.

Adaptation to the problem is the key to efficient solvers. Adaptation in many ways!

$O(n)$ reliable approximate solvers? A posteriori error analysis leading to efficient and reliable stopping criteria, including algebraic solvers ...... there will be price to pay!
Thank you for your kind patience!