Computation Lab - Nonlinear Pendulum

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Abstract

A computer program was written to model the simple pendulum. Cases of undamped, damped, and damped + driven were considered. In the two non-driven cases, comparisons were drawn between the cases of linear and non-linear modelling by comparing phase diagrams of each system for a series of initial conditions. In the case of the damped, driven pendulum, the effect of a control parameter was explored, namely the amplitude of the driving force. Period-doubling was demonstrated and it was found that for the value a=1.5, the system became chaotic.

Aims

To computationally model the following systems:

- a simple pendulum (linear and nonlinear)
- a damped pendulum (linear and nonlinear)
- a damped, driven nonlinear pendulum

Introduction and Theory

The most general form for describing a pendulum consisting of a bob at the end of a light inextensible string of length L is as follows:

$$\frac{d^2\theta}{dt^2} = -\beta^2 sin(\theta) - k\omega = acos(\Omega t)$$

In this equation, $\beta = \sqrt{\frac{g}{L}}$, k is a damping coefficient, a is the amplitude of the driving force and Ω its frequency. This can be specialised to the undamped or undriven cases by letting $k = 0$ or $a = 0$, respectively. It can be further specialised to the linear case by making the approximation $sin(\theta) \approx \theta$.

The Simple Euler Method

The *Simple Euler Method* uses Taylor series expansions for θ and ω, ignoring terms of order 2 or higher:

$$\theta(t + \Delta t) = \theta(t) + \dot{\theta}\Delta t + \dots$$

or, with a slight change in notation

$$\theta_{n+1} = \theta_n + \omega_n \Delta t$$

Similar analysis for ω yields:

$$\omega_{n+1} = \omega_n - \theta_n \Delta t$$

However, this is only accurate to first order, and thus not very useful for our particular purposes.

Trapezoidal Rule

An improvement on the simple Euler method, the trapezoidal rule calculates the are of the trapezoid between t and Δt:

$$\int_t^{t+\Delta t} \frac{d\theta}{dt} dt = \theta_{n+1} - \theta_n \approx ((t + \Delta t) - t)\frac{\frac{d\theta(t)}{dt} + \frac{d\theta(t+\Delta t)}{dt}}{2}$$

Which, taking a taylor expansion, gives

$$\theta_{n+1} \approx \theta_n + \frac{k1a + k2a}{2}$$

where

$$k1a = \omega_n \Delta t$$

$$k2a = (\omega_n + f(\theta_n, \omega_n, t)\Delta t)\Delta t$$

Similarly, we find

$$\omega_{n+1} \approx \omega_n + \frac{k1b + k2b}{2}$$

Where

$$k2b = (f(\theta_{n+1}, \omega_n + k1b, t_{n+1}))\Delta t$$

The Runge-Kutta Method

The fourth order Runge Kutta method is a more accurate approximation given by the following equations:

$$k1a = h\omega$$
$$k2a = h(\frac{\omega + k1b}{2})$$
$$k3a = h(\frac{\omega + k2b}{2})$$
$$k4a = h(\omega + k3b)$$

$$k1b = hf(\theta, \omega, t)$$
$$k2b = hf(\theta + \frac{k1a}{2}, \omega + \frac{k1b}{2}, t + \frac{h}{2})$$
$$k3b = hf(\theta + \frac{k2a}{2}, \omega + \frac{k2b}{2}, t + \frac{h}{2})$$
$$k4b = hf(\theta + k3a, \omega + k3b, t + h)$$
$$\theta(t + h) = \theta(t) + \frac{(k1a + 2k2a + 2k3a + k4a)}{6}$$
$$\omega(t + h) = \omega(t) + \frac{(k1b + 2k2b + 2k3b + k4b)}{6}$$

Experimental Method

Firstly, a C-program utilising the trapezoidal rule was used to create sets of data for both undamped and damped pendula in both the linear and non-linear case, for initial starting angles of 0.2, 1 and 3.124. Using Gnuplot, time-dependence graphs of angle and angular velocity were plotted, along with phase diagrams for each initial angle.

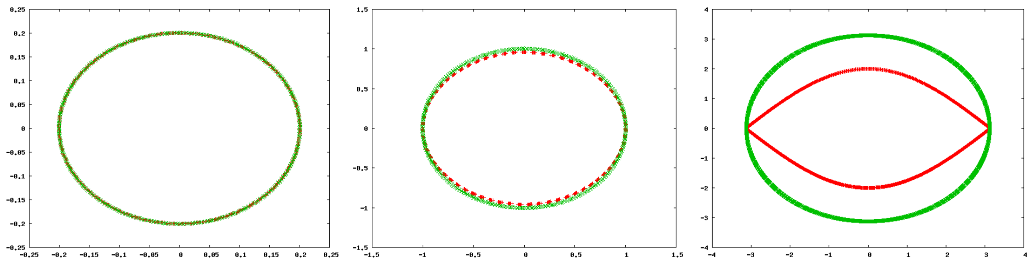
Next, using a modified C-program utilising the Runge-Kutta method, data was attained for the case of the damped, driven non-linear pendulum. The control parameter, a, was varied, corresponding to the amplitude of the driving force. The values of a for which phase portraits were plotted were $0.9, 1.07, 1.35, 1.47$ and 1.5. By plotting data only every period of the driving force, a Poincare section was produced for the case of $a = 1.5$

Finally, using a modified version of the above program, a bifurcation diagram was plotted of final periodic angles vs a.

Results and Analysis

Simple Pendulum

The trapezoidal rule was used to simulate the progress in time of a simple, undamped, non-driven pendulum. This was done twice, firstly using the linear approximation $sin(\theta) \approx \theta$, and then for the non-linear case. This was done for various initial starting angles θ. Phase-space diagrams were then plotted for each of these angles, with the linear case being plotted in green and the non-linear case plotted in red.

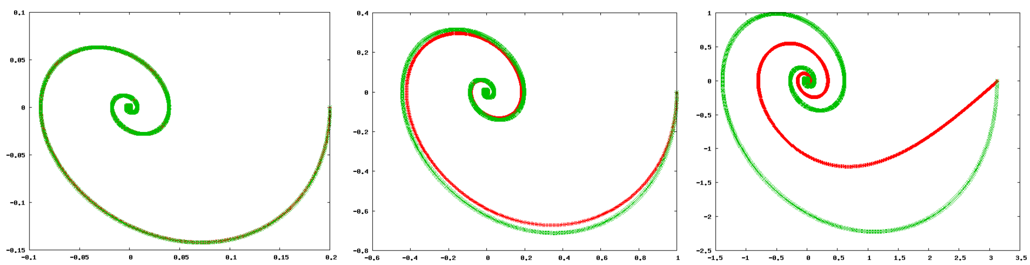


Phase portraits for (l-r), $\theta_i = 0.2$, $\theta_i = 1$, $\theta_i = 3.124$, for both linear (green) and nonlinear (red)

As we can see above, for small angle, the linear approximation is indeed a very good approximation. As the initial angle increases, we can see that the shape of the non-linear phase trajectory approaches that of the seperatrix.

Damped Pendulum

Using the Runge-Kutta method, the above steps were repeated for the case of a damped, undriven pendulum. The phase portraits are shown below.



Damped phase portraits for (l-r), $\theta_i = 0.2$, $\theta_i = 1$, $\theta_i = 3.124$, for both linear (green) and nonlinear (red)

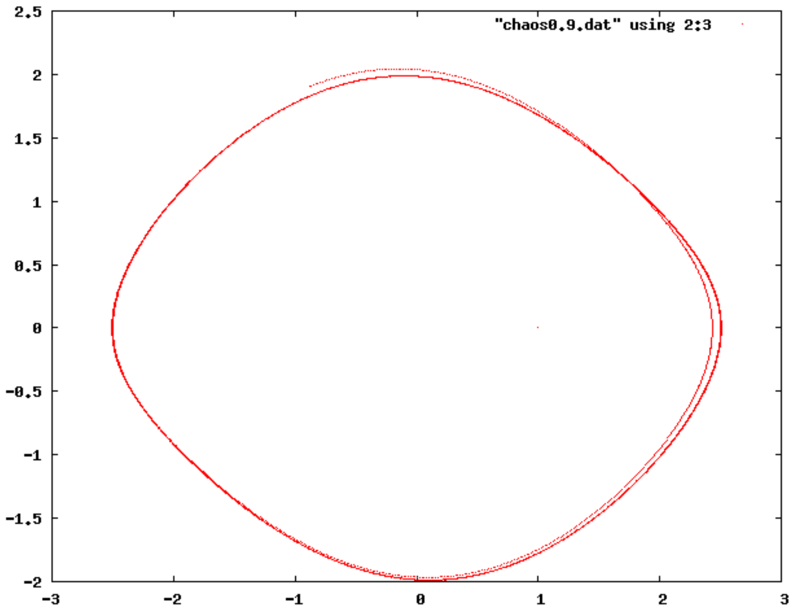
As we can see, for small starting angles, the linear case again agrees quite well for the non-linear case. For larger angles, we can see that even though initially the two equations yields quite different results, they eventually begin to agree after a sufficiently long time. This is even more evident in plots of angle vs time and angular velocity versus time, which are included in the appendix.

Damped Driven Pendulum

Again using the Runge-Kutta method, phase portraits were obtained for damped driven pendula with various driving amplitudes. In each case, the following parameters were fixed:

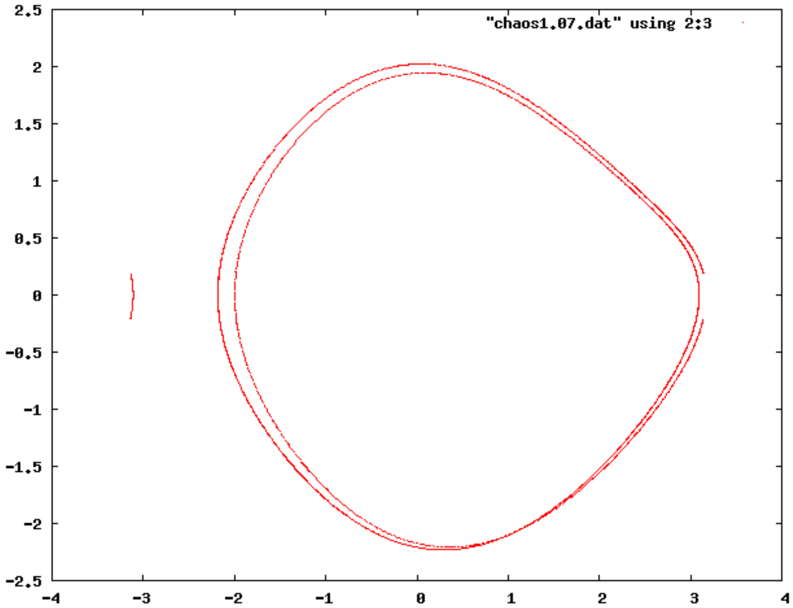
- $\theta_i = -2$
- $\omega_i = 0.5$
- $\beta = 0.5$

For a=0.9, we get normal pendulum behavior of a one-period closed phase trajectory. The tail at the top of the diagram shows a brief transition period before the motion settles into its periodic state.



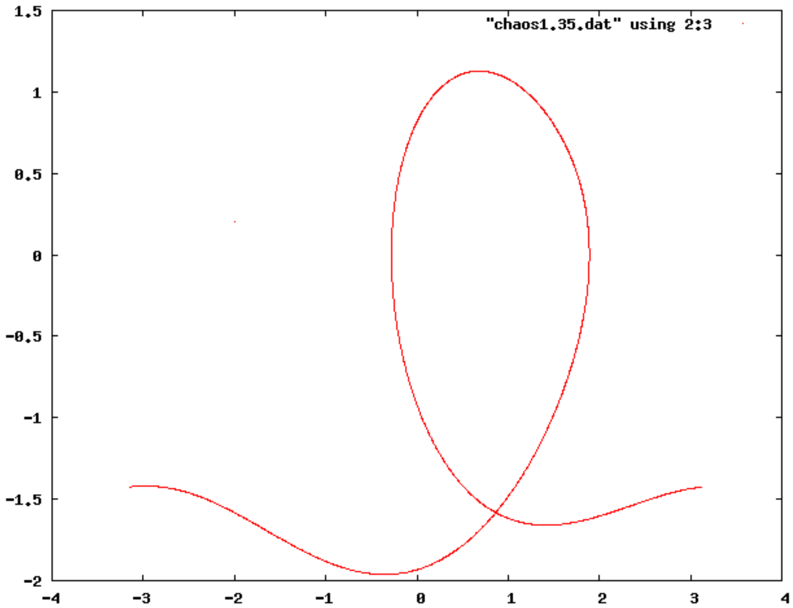
Phase portrait for $a = 0.9$

For a=1.07, we can see that period doubling has occurred, as evident from the two distinct loops.



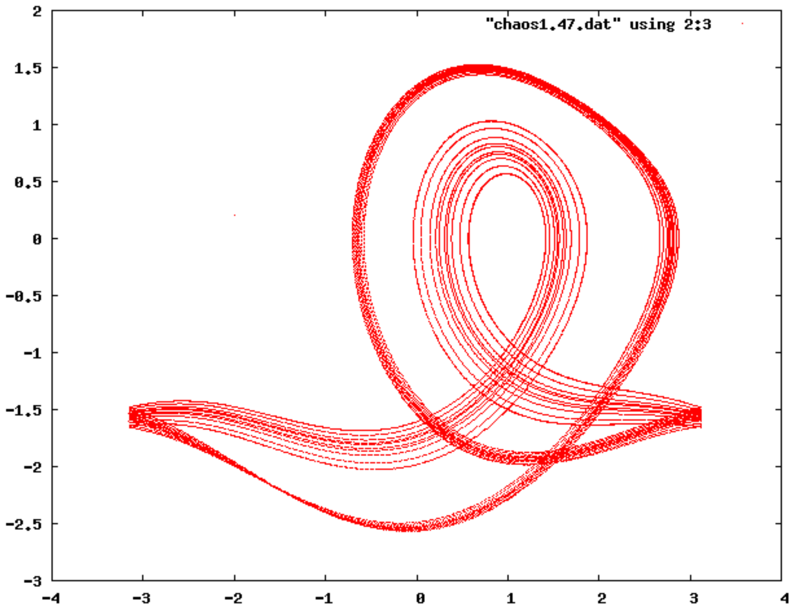
Phase portrait for $a = 1.07$

For a=1.35, we begin to get more complicated motion, owing to rotations beginning to occur.



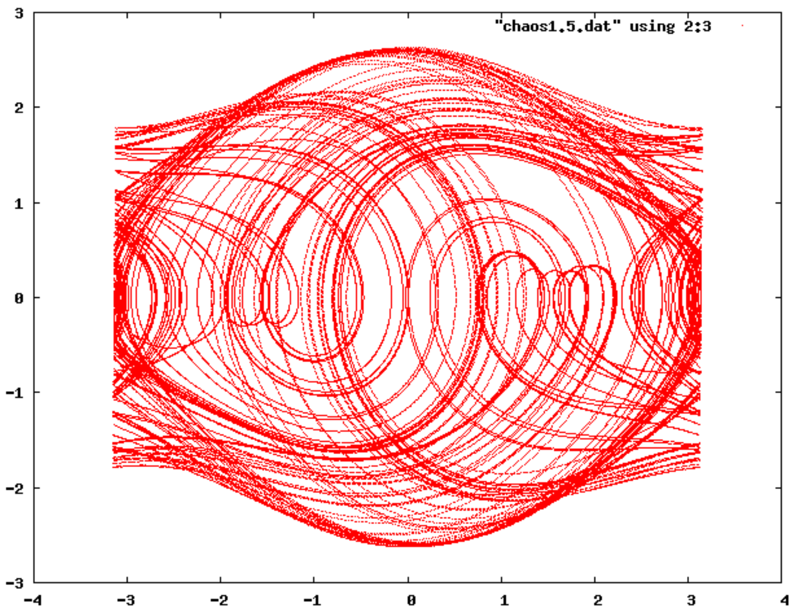
Phase portrait for $a = 1.35$

For a=1.47, we see that period doubling has occurred multiple times, leading to a complicated phase trajectory whose period is approximately ten times that of the driving force.



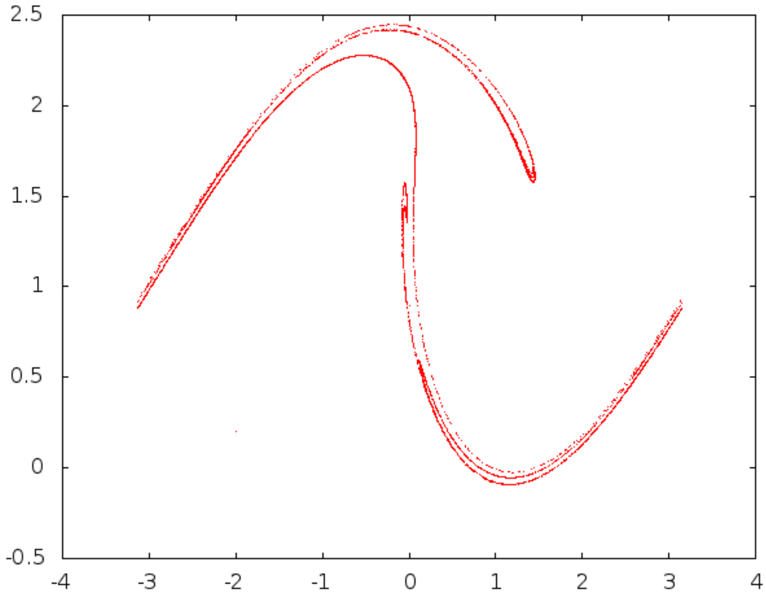
Phase portrait for $a = 1.47$

For a=1.5, we see chaotic motion.



Phase portrait for $a = 1.5$

One way of analysing this chaotic motion is by recording data once every period of the driving force, leading to the following Poincare Section:



Poincare Section for $a = 1.5$

All of this information can be neatly summarised in the following bifurcation diagram:

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In this diagram, we can clearly see the features outlined above. Of particular interest:

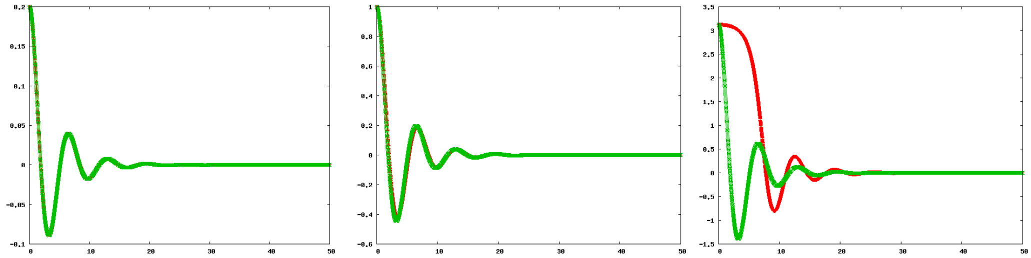
- At a=1.07 we can see the period doubling occurring as the bifurcation diagram branches off into two parts.
- Between a=1.15 and a=1.25 we can see dramatic period doubling and potentially chaos occurring. Although this region was not investigated greatly in the lab, a phase diagram for a=1.2 implied motion with an extremely long period. A phase diagram for a=1.26 showed multiple-period motion with rotations. These diagrams are included in the appendix.
- After a=1.3, we again get single-period motion, this time for the case where the pendulum makes full rotations.
- At a=1.42 we see period doubling starting again, ultimately leading to the chaotic region near a=1.5

Discussion and Conclusions

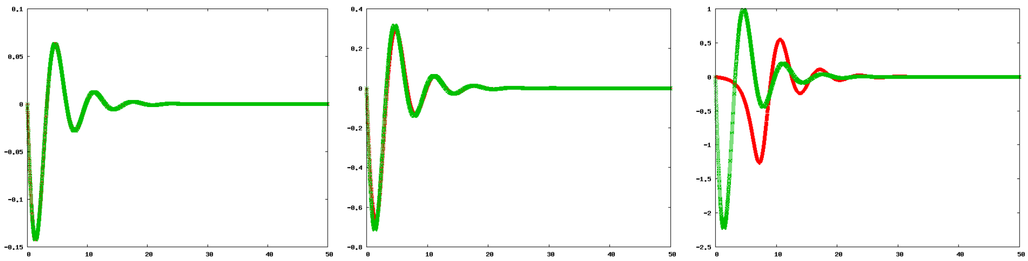
It was clearly seen that for large angles, the linear approximation of $sin(\theta) \approx \theta$ leads to large errors; this is the reason for most physical experiments on pendula requiring that all displacement angles be less than $10°$. It was also seen that in the cased of damped pendula, the linear and nonlinear cases tended to become close after a sufficient period of time had elapsed.

Using the Runge-Kutta method, the descent of a system into chaos through changes in a control parameter were clearly seen. Period doubling was also demonstrated. All of this information was neatly summarised in a bifurcation diagram.

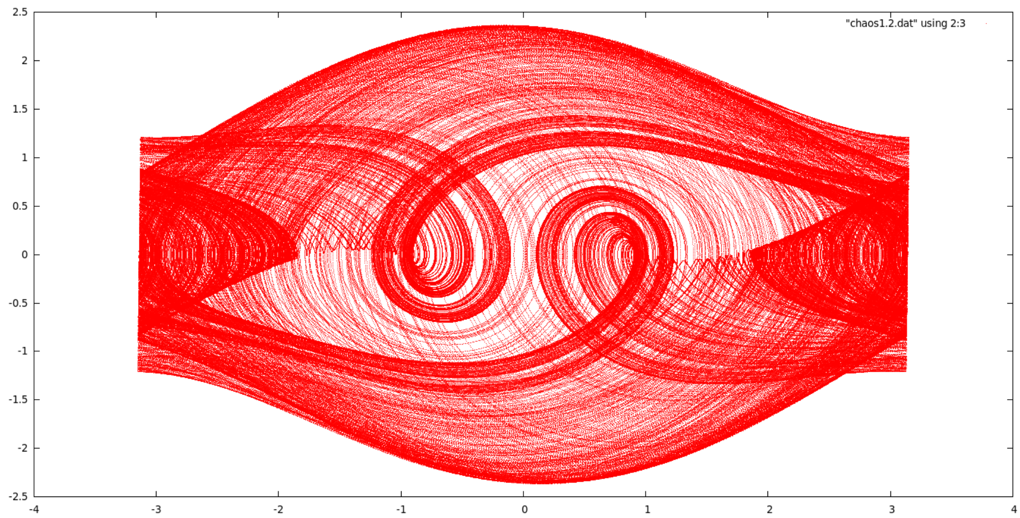
Appendix of diagrams



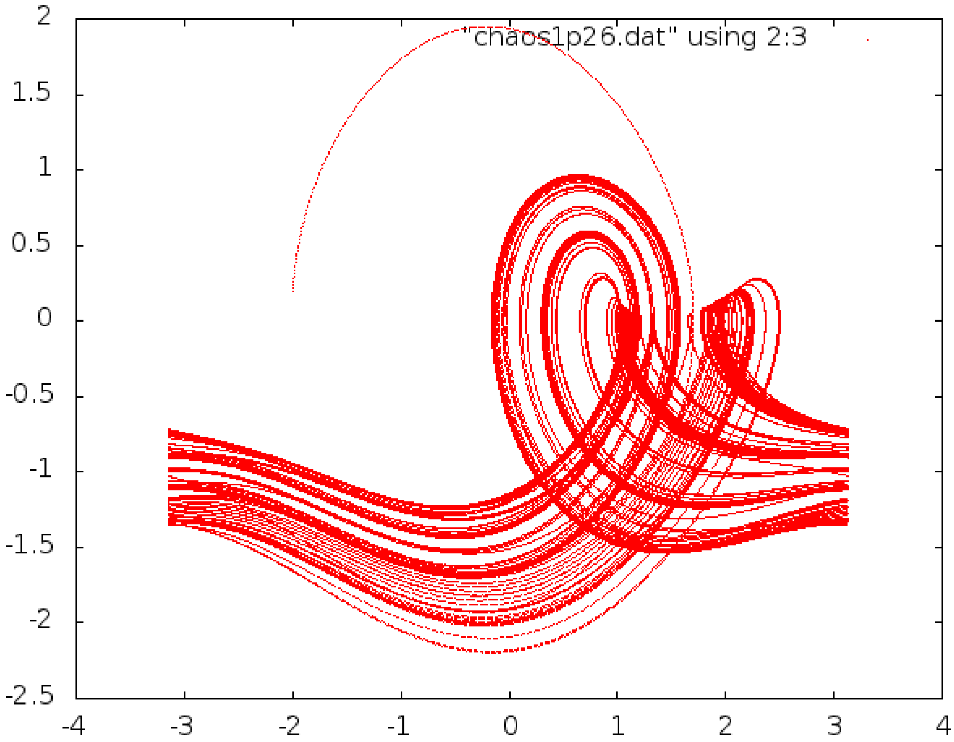
Graphs of θ vs time for (l-r), $\theta_i = 0.2$, $\theta_i = 1$, $\theta_i = 3.124$, for both linear (green) and nonlinear (red)



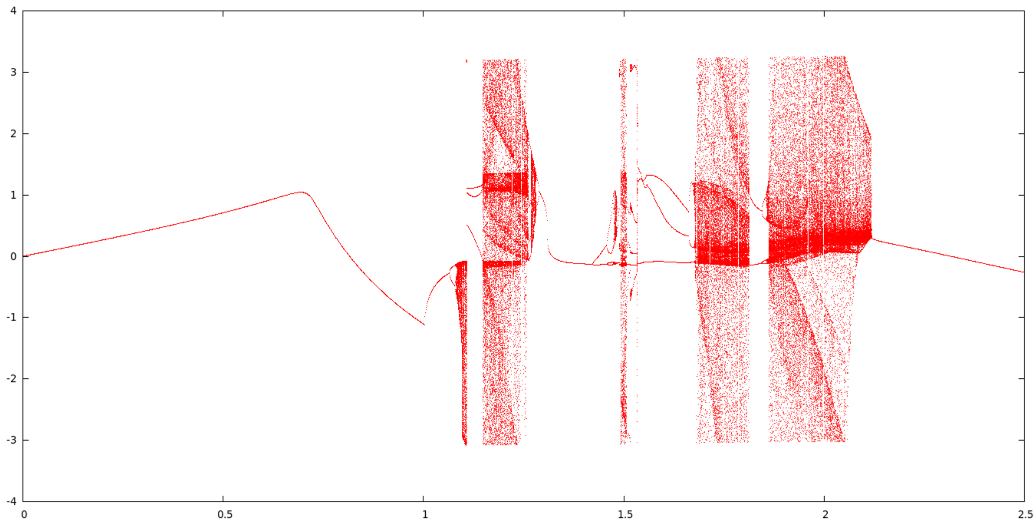
Graphs of ω vs time for (l-r), $\theta_i = 0.2$, $\theta_i = 1$, $\theta_i = 3.124$, for both linear (green) and nonlinear (red)



Phase portrait for $a = 1.2$



Phase portrait for $a = 1.26$



The bifurcation diagram, but this time with a wider range of a*, from* 0 *to* 2.5