INVOLUTION SUBWORD COMPLEXES IN COXETER GROUPS

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Abstract. Let (W, S) be a Coxeter system. An element in W and an ordered list of elements in S give us a subword complex, as defined by Knutson and Miller. We define the "fish product" between an involution in W and a generator in S, which is also discussed by Hultman. This "fish product" always produces an involution. The structure of the involution subword complex $\hat{\Delta}(Q, w)$ behaves very similarly to the regular subword complex. In particular, we prove that $\hat{\Delta}(Q, w)$ is either a ball or sphere. We then give an explicit description of a special class of involution subword complexes in S_n and demonstrate that they are isomorphic to the dual associahedron.

1. Introduction

Subword complexes were introduced by Knutson and Miller in [7] in order to discuss the combinatorics of determinantal ideal and Schubert polynomials, before being studied in their own right in [6]. There, it was realised that subword complexes also illuminated the nature of reduced expressions in Coxeter groups and had further connections to Grothendieck polynomials. One of their main results was a characterisation of the topology of these complexes. In brief: a subword complex $\Delta(Q, w)$ associated to a word Q in the generators of a Coxeter group, and an element w of the Coxeter group, has faces given by subwords $P \in Q$ such that $Q \setminus P$ is an expression for w. Topologically, it is shown that, regardless of choice of group, Q or w, $\Delta(Q, w)$ is always a sphere or a ball. Knutson and Miller also provide a method of determined the topology more precisely, by introducing the Demazure product, a maximal reduced subword of Q. By computing the Demazure product of Q, the topology becomes known.

Specific choices of Q and w were found to give quite interesting structures. By choosing lexicographically minimal subwords of c-sorting words in the symmetric group, Ceballos, Lambe and Stumpe were able to view the associahedron as a subword complex, establishing a correspondence between reduced expressions of w and triangulations in [2]. They went on to establish subword complexes which can be realised as generalised associahedra and multi-associahedra.

In [9] Pilaud and Stumpe extend this, working in more general groups to find the generalised associahedra of Hohlweg and Lange as subword complexes. The relationship between subword complexes and the associahedron is further strengthened by the work of Woo, who in [13] established a link between Schubert polynomials, the reason for the introduction of subword complexes, and the Catalan numbers, the vertex count of the associahedron.

In this paper, we develop the theory of subword complexes in a slightly different setting, introduced in [11] and developed by Hultman in [4]. We restrict ourselves to consider only involutions in our Coxeter group and introduce a new product in order to remain in this space.

For W a Coxeter group with generators S, $w \in W$ an involution, and $s \in S$, we define the "fish product"

$$w \rtimes s = \begin{cases} ws & \text{if } ws = sw \\ sws & \text{else} \end{cases}$$

With this product, we redefine the subword complex in Section 2 of this paper and go on to develop results analoguous to those of Knutson and Miller in Section 3. We find, once again, that the subword complex is topologically a ball or a sphere, and that it's topology is determined by an involution-Demazure product. We also note a few other properties, such as shellability and purity of the complex, that follow from the properties of the involution product.

In Section 4, we look at a family of subword complexes in the symmetric group., analoguous to those of Ceballos, Lambe and Stumpe, choosing an lexicographically minimal subword of an involution c-sorting word for complete reversal. We find that we obtain the Catalan numbers as a count for our facets and show that our complex is the dual associahedron. We establish a correspondence between triangulations of polygons and facets of our complex and provide a recursive construction of facets of our complex. We introduce a partial order on the facets, allowing us to consider them as points in the Tamari lattice.

In Section 5, we provide a generalisation of this family, which we believe to be analoguous to the work of Ceballos, Lambe and Stumpe, possibly leading to multi-associahedra. We mention a few preliminary results, which we hope will spark future work in this area

2. Topology of Subword Complexes

Throughout, let W be a Coxeter group with generators S. In this section, we present a result analogous to that of [6], but in the involution setting.

Definition 2.1. For $w \in W$ an involution, and $s \in S$, we define the "fish product" of w and s to be

$$w \rtimes s = \begin{cases} ws & \text{if } ws = sw \\ sws & \text{else} \end{cases}$$

Note that the result will once again be an involution.

Remark 2.2. For any $w \in W, s \in S$, we have that $(w \rtimes s) \rtimes s = w$. To see this, consider the two cases. First, where $ws = sw$. Then $(w \rtimes s) \rtimes s = (ws) \rtimes s$. Since $ws = sw$, we have that $wss = sws$ and so $ws \rtimes s = wss = w$. Second, where $ws \neq sw$. Then $(w \rtimes s) \rtimes s = (sws) \rtimes s$. Now, $(sws)s = sw \neq ws = s(sws)$, so $(sws) \rtimes s = s(sws)s = w$.

We will often abuse notation and write $w \rtimes s_1 \rtimes s_2 \cdots \rtimes s_n$ to indicate $(\cdots((w \rtimes s_1) \rtimes s_2) \cdots) \rtimes s_n$. If it is ever unclear, assume that we intend left associativity for the fish product.

Proposition 2.3. *In the fish product, the regular Coxeter relations hold i.e s, t commute if $m(s,t) = 2$, etc. In S_n, we also have $s \rtimes t \rtimes \cdots = t \rtimes s \rtimes \cdots$, for any $s, t \in S_n$*

Definition 2.4. A *word* of size m is an ordered sequence $Q = (s_1, s_2, \ldots, s_m)$ of elements of S. An ordered subsequence P of Q is called a *subword* of Q. We say that P *represents* $w \in W$ if the ordered fish product of the elements of P is an expression for w. We call such an expression *reduced* if it is an expression of minimal length. We say that P *contains* $w \in W$ if some subword of P represents w.

We denote by $\hat{\mathcal{R}}(w)$ the set of all words P such that P represents w. Also, let $\hat{\ell}(w)$ be the length of a reduced expression for w.

The *involution subword complex* $\hat{\Delta}(Q, w)$ is the set of subwords $Q \setminus P$ whose complements P contain an element of $\hat{\mathcal{R}}(w)$.

Remark 2.5. $\hat{\mathcal{R}}(w)$ is closed under the Coxeter relations. That is, any expression related to an element of $\hat{\mathcal{R}}(w)$ by a finite application of the Coxeter relations is also an element of $\hat{\mathcal{R}}(w)$.

Definition 2.6. A *k-simplex* is a k-dimensional polytope that is the convex hull it $k+1$ vertices.

Definition 2.7. A *simplicial complex* K is a collection of simplices such that
1. Any face of a simplex in K is also in K.
2. The intersection of any two simplices $\sigma_1, \sigma_2 \in K$ is a face of both σ_1 and σ_2.

The following is immediate from the definitions and the fact that all reduced expressions for a given element have the same length.

Lemma 2.8. *$\hat{\Delta}(Q, w)$ is a pure simplicial complex whose facets are the subwords $Q \setminus P$ such that $P \subseteq Q$ represents w.*

Proposition 2.9 (Exchange Property - [4])**.** *Suppose $Q = (s_1, \ldots, s_k)$ represents w and $\hat{\ell}(w \rtimes s) < k$ for some $s \in S$. Then $w \rtimes s = s_1 \rtimes \cdots \rtimes \tilde{s}_i \rtimes \cdots \rtimes s_k$ for a unique $i \in [k]$, where $\tilde{s}_i$ denotes the deletion of s_i.*

Proposition 2.10 (Lifting Property - [4])**.** *Let $s \in S$ and $v, w \in W$ satisfy $v \leq w$ and $w \rtimes s \leq w$. Then $v \rtimes s \leq w$.*

Definition 2.11. Let Δ be a simplicial complex and let $v \in \Delta$ be a single vertex.
1. The *deletion* of v from Δ is $\mathrm{del}(v, \Delta) = \{F \in \Delta : v \notin F\}$.
2. The *link* of v in Δ is $\mathrm{link}(v, \Delta) = \{F \setminus \{v\} : F \in \Delta, v \in F\}$.
3. Δ is *vertex decomposable* if Δ is pure (all facets of Δ are the same size) and either
   i) $\Delta = \{\emptyset\}$ or
   ii) For some vertex $v \in \Delta$, both $\mathrm{del}(v, \Delta)$ and $\mathrm{link}(v, \Delta)$ are vertex-decomposable.
4. A *shelling* of Δ is an ordered list $F_1, F_2, \ldots, F_k$ of the facets of Δ such that for each $i \leq t$, $\bigcup_{j<i} \overline{F_j} \cap \overline{F_i}$ is a subcomplex generated by codimension 1 faces of F_i, where $\overline{F}$ denotes the set of faces of F.
5. Δ is *shellable* if it is pure and has a shelling.

It was proved by Provan and Billera [10] that vertex-decomposability implies shellability, so in order to establish shellability of the involution subword complexes, we only need to prove that they are vertex-decomposable.

Lemma 2.12. *$\hat{\Delta}(Q,w)$ is vertex-decomposable, hence shellable.*

Proof. We proceed by induction on the length of Q. Let $Q = s_1 s_2 \cdots s_k$ and let $Q' := s_1 s_2 \cdots s_{k-1}$. We claim $\mathrm{link}(s_k, \hat{\Delta}(Q,w)) = \hat{\Delta}(Q',w)$ and $\mathrm{del}(s_k, \hat{\Delta}(Q,w)) = \hat{\Delta}(Q',w)$ or $\hat{\Delta}(Q', w \rtimes s_k)$.

Suppose $P \in \mathrm{link}(s_k, \hat{\Delta}(Q,w))$ Then $P \cup \{s_k\} \in \hat{\Delta}(Q,w)$. Then $Q \setminus (P \cup \{k\})$ contains an element of $\hat{\mathcal{R}}(w)$ and hence $Q' \setminus P$ contains an element of $\hat{\mathcal{R}}(w)$ and $P \in \hat{\Delta}(Q',w)$. Similarly, every element of $\hat{\Delta}(Q',w)$ is an element of $\mathrm{link}(s_k, \hat{\Delta}(Q,w))$ and so $\mathrm{link}(s_k, \hat{\Delta}(Q,w)) = \hat{\Delta}(Q',w)$.

Suppose $P \in \mathrm{del}(s_k, \hat{\Delta}(Q,w))$ and no element of $\hat{\mathcal{R}}(w)$ ends in s_k. Then, $P \in \hat{\Delta}(Q,w)$, $s_k \in Q \setminus P$ and $Q \setminus P$ contains an element of $\hat{\mathcal{R}}(w)$. Hence $Q' \setminus P$ contains an element of $\hat{\mathcal{R}}(w)$ and so $P \in \hat{\Delta}(Q',w)$. Similarly, every element of $\hat{\Delta}(Q',w)$ is an element of $\mathrm{del}(s_k, \hat{\Delta}(Q,w))$ and so $\mathrm{del}(s_k, \hat{\Delta}(Q,w)) = \hat{\Delta}(Q',w)$.

If we have an element of $\hat{\mathcal{R}}(w)$ ending in s_k, then we claim $\mathrm{del}(s_k, \hat{\Delta}(Q,w)) = \hat{\Delta}(Q', w \rtimes s_k)$. It is clear that every element of $\hat{\Delta}(Q', w \rtimes s_k)$ is an element of $\mathrm{del}(s_k, \hat{\Delta}(Q,w))$. Suppose $P \in \mathrm{del}(s_k, \hat{\Delta}(Q,w))$. Then $P \in \hat{\Delta}(Q,w)$ and $s_k \in Q \setminus P$. If $Q \setminus P$ contains an element of $\hat{\mathcal{R}}(w)$ ending in s_k, then $Q' \setminus P$ contains an element of $\hat{\mathcal{R}}(w \rtimes s_k)$. If $Q \setminus P$ contains an element of $\hat{\mathcal{R}}(w)$ not ending in s_k, then $Q' \setminus P$ contains an element of $\hat{\mathcal{R}}(w \rtimes s_k)$, by the exchange relation. Hence $P \in \hat{\Delta}(Q', w \rtimes s_k)$ and so $\mathrm{del}(s_k, \hat{\Delta}(Q,w)) = \hat{\Delta}(Q', w \rtimes s_k)$.

Therefore, by induction, both $\mathrm{link}(s_k, \hat{\Delta}(Q,w))$ and $\mathrm{del}(s_k, \hat{\Delta}(Q,w))$ are vertex-decomposable. Thus, $\hat{\Delta}(Q,w)$ is vertex-decomposable.

□

Definition 2.13. We define the *Demazure product* of $w \in W$ and $s \in S$ by

$$w \circ s = \begin{cases} w \rtimes s & \text{if } \hat{\ell}(w \rtimes s) > \hat{\ell}(w) \\ w & \text{if } \hat{\ell}(w \rtimes s) < \hat{\ell}(w) \end{cases}$$

We define the *Demazure product* of a word $Q = s_1 s_2 \cdots s_k$ to be $\hat{\delta}(Q) = (\cdots((s_1 \circ s_2) \circ s_3) \cdots \circ) \circ s_k$

Notation: If some reduced expression of $v \in W$ has a subword that represents $w \in W$, we write $v \geq w$. This is the involution analogue of the Bruhat partial order. If a *proper* subword of a reduced expression of v represents w, we write $v > w$.

For a word $Q = (s_1, \ldots, s_k)$ in S and $s \in S$, we write $s \rtimes Q$ to indicate the letter-by-letter product of s and Q, $s \rtimes s_1 \rtimes \cdots \rtimes s_k$. Similarly, we have $Q \rtimes s = s_1 \rtimes \cdots \rtimes s_k \rtimes s$. Given a second word $P = (t_1, \ldots, t_r)$ in S, we write $Q \rtimes P$ to indicate $s_1 \rtimes \cdots \rtimes s_k \rtimes t_1 \rtimes \cdots \rtimes t_r$. We will use $\overleftarrow{Q}$ to denote the word formed by reversing the letters of Q.

Lemma 2.14. *Let P be a word in W and $w \in W$.*

(1) The Demazure product $\hat{\delta}(P) \geq w$ if and only P contains w.

(2) If $\hat{\delta}(P) = w$, then every subword of P containing w has Demazure product w.
(3) If $\hat{\delta}(P) > w$, then P contains a word T representing an element $w' > w$ satisfying $|T| = \hat{\ell}(w') = \hat{\ell}(w) + 1$.

Proof. (1) First, suppose that $\hat{\delta}(P) \geq w$.
Let $w' = \hat{\delta}(P)$, and let $P' \subseteq P$ be the subword obtained by reading P in order and omitting any generators that do not increase length. Notice that P' represents w' by construction. Since $w' \geq w$, we have by definition that any reduced expression for w' has a subword that represents w. Thus, since P' is a reduced expression for w', P' has a subword that represents w. Hence, P' contains w and thus P also contains w.

Now, suppose that P contains w.
Let $T \subseteq P$ such that T represents w. We will induct on $|P|$. Let $s \in S$ be the last generator in the word P. Then, from the definition of the Demazure Product and the exchange property we have that $\hat{\delta}(P) \rtimes s < \hat{\delta}(P)$. Also, by the definition of the Demazure Product, we have that $\hat{\delta}(P \setminus s)$ is $\hat{\delta}(P)$ or $\hat{\delta}(P) \rtimes s$. Hence, $\hat{\delta}(P \setminus s) \leq \hat{\delta}(P)$.
First, consider the case where $w \rtimes s > w$. This implies that T cannot have s as its final letter, so $T \subseteq P \setminus s$. By induction we have that $w \leq \hat{\delta}(P \setminus s)$. We have from above that $\hat{\delta}(P \setminus s) \leq \hat{\delta}(P)$, so $w \leq \hat{\delta}(P)$ as desired.
Now, consider the case where $w \rtimes s < w$. Let $T' \subset T$ such that T' represents $w \rtimes s$. Again we have that T' cannot have s as its final letter, so $T' \subseteq P \setminus s$. So by induction we have that $w \rtimes s \leq \hat{\delta}(P \setminus s)$. Since $w \rtimes s < w$ and $w \rtimes s \leq \hat{\delta}(P \setminus s) \leq \hat{\delta}(P)$, by the lifting property, $w \leq \hat{\delta}(P)$.

(2) Let $P' \subseteq P$ be such that P' contains w. We have from the definition of the Demazure Product that $\hat{\delta}(P) \geq \hat{\delta}(P')$. Since part 1 holds, we have that $\hat{\delta}(P') \geq w$. By assumption, $\hat{\delta}(P) = w$. Hence, $w = \hat{\delta}(P) \geq \hat{\delta}(P') \geq w$ and thus $\hat{\delta}(P') = w$.

(3) Choose any $w' \in W$ such that $\hat{\ell}(w') = \hat{\ell}(w) + 1$ and $w < w' \leq \hat{\delta}(P)$, then apply part 1 to conclude that P contains a word T representing w'.

□

Lemma 2.15. *Let T be a word in W, and $w \in W$ such that $|T| = \hat{\ell}(w) + 1$.*
(1) There are at most two elements $s \in T$ such that $T \setminus s$ represents w.
(2) If $\hat{\delta}(T) = w$,then there are two distinct $s \in T$ such that $T \setminus s$ represents w.
(3) If T represents $w' > w$, then $T \setminus s$ represents w for exactly one $s \in T$.

Proof. (1) It is trivial if $|T| \leq 2$, so assume $|T| \geq 3$ and that there exist $s_1, s_2, s_3 \in T$ such that $T \setminus s_i$ represents w for $i = 1, 2, 3$. Let $T = T_1 s_1 T_2 s_2 T_3 s_3 T_4$. Since $T \setminus s_2$ and $T \setminus s_1$ both represent w, we have that

$$w = id \rtimes T_1 \rtimes s_1 \rtimes T_2 \rtimes T_3 \rtimes s_3 \rtimes T_4 = id \rtimes T_1 \rtimes T_2 \rtimes s_2 \rtimes T_3 \rtimes s_3 \rtimes T_4.$$

We can cancel from the right to get $id \rtimes T_1 \rtimes s_1 \rtimes T_2 = id \rtimes T_1 \rtimes T_2 \rtimes s_2$, which implies

$$id = id \rtimes T_1 \rtimes T_2 \rtimes s_2 \rtimes \overleftarrow{T_2} \rtimes s_1 \rtimes \overleftarrow{T_1}. \tag{1}$$

Since $T \setminus s_3$ represents w, we have that

$$w = id \rtimes T_1 \rtimes s_1 \rtimes T_2 \rtimes s_2 \rtimes T_3 \rtimes T_4.$$

Substituting the expression for id obtained in (1),

$$\begin{aligned} w &= id \rtimes T_1 \rtimes T_2 \rtimes s_2 \rtimes \overleftarrow{T_2} \rtimes s_1 \rtimes \overleftarrow{T_1} \rtimes T_1 \rtimes s_1 \rtimes T_2 \rtimes s_2 \rtimes T_3 \rtimes T_4 \\ &= T_1 \rtimes T_2 \rtimes T_3 \rtimes T_4. \end{aligned}$$

Therefore $T \setminus s_3$ was not reduced, giving a contradiction.

(2) $\hat{\delta}(T) = w$ means that there is some $s \in T$ such that $T = T_1 s T_2$ and $T_1 T_2$ represents w. Furthermore, if T_1 represents w_1, then $\hat{\delta}(T_1) = \hat{\delta}(T_1 s) = w_1$ and so $\hat{\ell}(w_1) > \hat{\ell}(w_1 \rtimes s)$. Hence $w_1 > w_1 \rtimes s$ by the exchange relation. This means that omitting some $s' \neq s$ from T_1 gives a reduced expression for $w_1 \rtimes s$ and thus both $T \setminus s$ and $T \setminus s'$ represent w.

(3) This follows immediately from the exchange property.

□

We use the above lemmas to discern the topology of our complex, alongside the following result:

Lemma 2.16. *Suppose every codimension 1 face of a shellable simplicial complex Δ is contained in at most two facets. The Δ is a topological manifold-with-boundary that is homeomorphic to a either a ball or a sphere. The facets of the topological boundary of Δ are the codimension 1 faces of Δ contained in exactly one facet of Δ*

Proof. See [1], Proposition 4.7.22 □

Theorem 2.17. *The subword complex $\hat{\Delta}(Q, w)$ is either a ball or a sphere. A face $Q \setminus P$ is in the boundary of $\hat{\Delta}(Q, w)$ if and only if P has Demazure product $\hat{\delta}(P) \neq w$.*

Proof. By part 1 of Lemma 2.15, every codimension 1 face of $\hat{\Delta}(Q, w)$ is contained in at most two facets, and $\hat{\Delta}(Q, w)$ is shellable by Lemma 2.12. Hence, by Lemma 2.16, $\hat{\Delta}(Q, w)$ is homeomorphic to a ball or sphere.

If $Q \setminus P$ is a face and $\hat{\delta}(P) \neq w$, then, by part 1 of Lemma 2.14, $\hat{\delta}(P) > w$ and so we can choose T as in part 3 of Lemma 2.14. Then, by part 3 of Lemma 2.15 $Q \setminus T$ is a codimension 1 face contained in exactly one facet of $\hat{\Delta}(Q, w)$. Thus, by Lemma 2.15, $Q \setminus P \subset Q \setminus T$ is in the boundary of $\hat{\Delta}(Q, w)$.

If $\hat{\delta}(P) = w$, then part 2 of Lemmas 2.14 and 2.15 tell us that every codimension 1 face $Q \setminus T \in \hat{\Delta}(Q, w)$ containing $Q \setminus P$ is contained in two facets of $\hat{\Delta}(Q, w)$. Lemma 2.16 says that every such $Q \setminus T$, and hence $Q \setminus P$, is in the interior of $\hat{\Delta}(Q, w)$.

□

Corollary 2.18. *The complex $\hat{\Delta}(Q, w)$ is a sphere if $\hat{\delta}(Q) = w$ and a ball otherwise*

3. Catalan Subword Complexes

3.1. **Background.** Recall that the symmetric group S_n is a Coxeter group with generators $s_i = (i\ i{+}1),\ i = 1, 2, \ldots n-1$. In this section, we let our Coxeter group to be S_{2n} and we use i to denote s_i.

Let w_0 be the reverse permutation in S_{2n}, given in one-line notation as $2n\ 2n{-}1\dots 1$. We will be interested in the word $1\ 2\dots 2n\ 2\ 3\dots 2n{-}1\dots n\ n{+}1$ in the generators of S_{2n}, which we denote by Q_n. This word represents w_0 and is in fact the lexicographically minimal such word. Similarly, let $w_0{}'$ be the reverse permutation in S_{2n-1}. We will be interested in the word $1\ 2\dots 2n{-}1\ 2\ 3\dots 2n{-}2\dots n$, denoted Q'_n. This is a word of interest, as it is the lexicographically minimal involution word of the reverse permutation, and is thus an involution analogue of the **c-sorting word** for the reverse permutation with $c = 1\ 2\ 3\dots 2n{-}1\ 2n$

Let the *blocks* be the longest consecutive increasing subwords of Q_n and Q'_n. So the first block of Q_n is $1\ 2\dots 2n$, the second block is $2\ 3\dots 2n{-}1$, and so on. Similarly, the first block of Q'_n is $1\ 2\dots 2n{-}1$, the second block is $2\ 3\dots 2n{-}2$,

Definition 3.1. Let $\hat{\Delta}_n$ be $\hat{\Delta}(Q_n, w_0)$ and let F_n be the number of facets of $\hat{\Delta}_n$.
Let $\hat{\Delta}'_n = \hat{\Delta}(Q'_n, w'_0)$.

We will show that the complexes $\hat{\Delta}_n$ and $\hat{\Delta}_n{}'$ each have C_n facets, where C_n is the n^{th} Catalan number, given by $\frac{1}{n+1}\binom{2n}{n}$.

Lemma 3.2. $Q = (s_1, s_2, \dots, s_{2n-1}, s_2, s_3, \dots, s_{2n-2}, \dots, s_n) \in \hat{\mathcal{R}}(w_0)$.

Proof. We know from [5] that the length of any reduced word of w_0 is n^2, so we just need to find an expression for w_0 of n^2 terms. We claim that

$$w_0 = s_1 \rtimes s_2 \rtimes \cdots \rtimes s_{2n-1} \rtimes s_2 \rtimes s_3 \rtimes \cdots \rtimes s_{2n-2} \rtimes \cdots \rtimes s_n.$$

First, consider the value of $s_i \rtimes s_{i+1} \rtimes \cdots \rtimes s_j$ for some $i < j$. Since $s_i s_{i+1} \neq s_{i+1} s_i$, we have $(s_{i+1} s_i s_{i+1}) \rtimes s_{i+2} \rtimes \cdots \rtimes s_j$. Now $s_{i+1} s_i s_{i+1}$ is the transposition $(i\ i{+}2)$. Notice that, in general, $s_{i+k}(i\ i{+}k) \neq (i\ i{+}k)s_{i+k}$, so to compute $(i\ i{+}k) \rtimes s_{i+k}$, we will conjugate to get the transposition $(i\ i{+}k{+}1)$. By induction, we have that $s_i \rtimes s_{i+1} \rtimes \cdots \rtimes s_j$ is the transposition $(i\ j+1)$.
Notice also that once we have finished expanding the section $s_i \rtimes s_{i+1} \rtimes \cdots \rtimes s_j$, none of the following terms will involve either i or $j + 1$, since they are all of the form s_k for $i < k < j$. Hence, the transposition $(i\ j{+}1)$ will commute with those terms since they form disjoint cycles. Thus, we can ignore the $(i\ j{+}1)$ when computing the remainder of the expression.
So, we have

$$s_1 \rtimes s_2 \rtimes \cdots \rtimes s_{2n-1} \rtimes s_2 \rtimes s_3 \rtimes \cdots \rtimes s_{2n-2} \rtimes \cdots \rtimes s_n = (1\ 2n)(2\ 2n{-}1)\cdots(n{-}1\ n{+}2)(n\ n{+}1),$$

which can be easily seen to be an expression for w_0. $\square$

Remark 3.3. Every facet of the complex $\hat{\Delta}_n$ contains the vertex $2n$.

Lemma 3.4. *Only even digits appear as vertices and hence we have at most* $\binom{n+1}{2}$ *vertices.*

Proof. To see this, first note that, due to the block structure of our word, we only need to consider the Demazure product on pairs of blocks. Next note that

$$\hat{\delta}(k\ k{+}1\dots l{-}1\ l\ k\ k{+}1\dots l{-}1\ l) = \hat{\delta}(k\ k{+}1\dots l{-}1\ l\ k\ k{+}1\dots l{-}1)$$

when $l > k$, as

$$\begin{aligned}
w &= s_k \rtimes s_{k+1} \rtimes \cdots \rtimes s_{l-1} \rtimes s_l \rtimes s_k \rtimes s_{k+1} \rtimes \cdots \rtimes s_{l-1} \rtimes s_l \\
&= s_k \rtimes s_{k+1} \rtimes s_k \rtimes s_{k+2} \rtimes s_{k+1} \rtimes \cdots \rtimes s_{l-2} \rtimes s_l \rtimes s_{l-1} \rtimes s_l \text{ by the Coxeter relations} \\
&= s_{k+1} \rtimes s_k \rtimes s_{k+2} \rtimes s_{k+1} \rtimes \cdots \rtimes s_{l-2} \rtimes s_{l-1} \rtimes s_l \rtimes s_{l-1} \rtimes s_l \text{ by the Coxeter relations} \\
&= s_{k+1} \rtimes \cdots \rtimes s_l \rtimes s_{l-1} \rtimes s_l \rtimes s_l
\end{aligned}$$

Similarly

$$\hat{\delta}(1\ 2 \ldots 2n{-}1\ 2n) = \hat{\delta}(1\ 2 \ldots 2n{-}1)$$

$$\hat{\delta}(k\ k{+}1 \ldots k{+}i\ k{+}i{+}2 \ldots l\ k{+}1 \ldots k{+}i) = \hat{\delta}(k\ k{+}1 \ldots k{+}i\ k{+}i{+}2 \ldots l\ k{+}1 \ldots k{+}i{-}1)$$

$$\hat{\delta}(k\ k{+}1\ k{+}2 \ldots l\ k\ k{+}1 \ldots l{-}1) = \hat{\delta}(k\ k{+}1\ k{+}2 \ldots l\ k{+}1 \ldots l{-}1)$$

Let $Q' = (1\,2 \ldots 2n\ 2 \ldots n\ n{+}1)$, omitting the K^{th} letter, and suppose K is odd. If it is in the k^{th} block, the Demazure product up to that block will give $w = (1\ 2n)(2\ 2n{-}1) \cdots (l{-}1\ 2n{-}l{+}2)$. If the K^{th} letter is $k + i$ and not at the start of the block, i.e $i \neq 0$, then we can multiply through the next i blocks to get

$$\hat{\delta}(Q') = \hat{\delta}(w\, k \cdots k{+}i{-}1\ k{+}i{+}1 \cdots\ 2n{-}k{+}1\ k{+}1 \cdots\ k{+}i{-}2\ k{+}i \cdots 2n{-}(k{+}i{-}1){+}1\ k{+}i{+}i \cdots)$$

omitting one letter from each of the blocks starting with k through to $k+i$. Then the $K+i^{th}$ block will be missing the first letter, since the first letter of each block is in an odd position and the letter we eliminate has the same parity as the missing one in the previous block ($i + (2n - k) - (k + i - 1)$ is even), and as we move one letter closer to the start of the block as we move from block to block, it must happen.In that block, we must also omit the last letter, as we could otherwise obtain the sequence $l{-}1\ l\ l{-}1\ l$, and, hence we must have that every block afterwards is missing the last letter. We have omitted at least $n+1$ letters. Thus, $\hat{\ell}(\hat{\delta}(Q')) < n^2$ and so $\hat{\delta}(Q')$ doesn't contain w_0. Hence, neither does Q'. So odd digits do not appear as vertices, giving our result.

□

Lemma 3.5. $(k\ k{+}1)(k{+}2\ k{+}3) \cdots (k{+}l\ k{+}l{+}1) \rtimes s_{k+1} \rtimes s_{k+2} \rtimes \cdots \rtimes s_{k+l} =$ $(k\ k{+}l{+}1)(k{+}1\ k{+}2)(k{+}3\ k{+}4) \cdots (k{+}l{-}1\ k{+}l)$ *for all even l*

Proof. We proceed by induction. Let $w = (k\ k{+}m)(k{+}1\ k{+}2)(k{+}3\ k{+}4) \ldots (k{+}m{-}2\ k{+}m{-}1)$. Note that

$$\begin{aligned}
w \rtimes s_{k+m} \rtimes s_{k+m+1} =& (k\ k{+}m)(k{+}1\ k{+}2)(k{+}3\ k{+}4) \cdots \\
& (k{+}m{-}2\ k{+}m{-}1)(k{+}m{+}1\ k{+}m{+}2) \rtimes s_{k+m} \rtimes s_{k+m+1} \\
=& (k{+}m\ k{+}m{+}1)(k\ k{+}m)(k{+}1\ k{+}2)(k{+}3\ k{+}4) \cdots (k{+}m{-}2\ k{+}m{-}1) \\
& (k{+}m\ k{+}m{+}1) \rtimes s_{k+m+1} \\
=& (k{+}m{+}1\ k{+}m{+}2)(k\ k{+}m{+}1)(k{+}1\ k{+}2)(k{+}3\ k{+}4) \cdots \\
& (k{+}m{-}2\ k{+}m{-}1)(k{+}m\ k{+}m{+}2)(k{+}m{+}1\ k{+}m{+}2) \\
=& (k\ k{+}m{+}2)(k{+}1\ k{+}2)(k{+}3\ k{+}4) \cdots (k{+}m{-}2\ k{+}m{-}1) \\
& (k{+}m\ k{+}m{+}1)
\end{aligned}$$

Then, as $(k\ k{+}1)(k{+}2\ k{+}3) \rtimes s_{k+1} \rtimes s_{k+2} = (k\ k{+}3)(k{+}1\ k{+}2)$, we can iterate this to obtain the result. □

Lemma 3.6. *$\{2,4,6,\ldots,2n\}$ is a facet of $\hat{\Delta}_n$*

Proof. $s_1 \rtimes s_3 \rtimes s_5 \rtimes \cdots \rtimes s_{2n-1} = (1\ 2)(3\ 4)\cdots(2n{-}2\ 2n{-}1)$ and so by Lemma 3.5

$$s_1 \rtimes s_3 \rtimes s_5 \rtimes \cdots \rtimes s_{2n-1} \rtimes s_2 \rtimes \cdots \rtimes s_{2n-1} = (1\ 2n)(2\ 3)(4\ 5)\cdots(2n{-}2\ 2n{-}1)$$

Repeatedly applying Lemma 3.5 in this manner, we get

$$s_1 \rtimes s_3 \rtimes \cdots \rtimes s_{2n-1} \rtimes s_2 \rtimes s_3 \rtimes \cdots \rtimes s_{2n-1} \rtimes \cdots \rtimes s_n \rtimes s_{n+1} = (1\ 2n)(2\ 2n{-}1)\cdots(n\ n{+}1).$$

Hence $s_1 \rtimes s_3 \rtimes \cdots \rtimes s_{2n-1} \rtimes s_2 \rtimes s_3 \rtimes \cdots \rtimes s_{2n-1} \rtimes \cdots \rtimes s_n \rtimes s_{n+1} \in \hat{\mathcal{R}}(w)$, giving our result. □

3.2. **Counting Facets.**

Theorem 3.7. *Let F_n be the number of facets of $\hat{\Delta}_n$. Then $F_{n+1} = \sum_{i=0}^{n} F_i F_{n-i}$*

In order to show this, we will establish a bijection between the facets of $\hat{\Delta}_{n+1}$ and pairs of facets in $\hat{\Delta}_i$ and $\hat{\Delta}_{n-i}$ for $0 \le i \le n$. We define $\hat{\Delta}_0$ to be the complex with exactly one facet, namely, the empty facet.

For a facet F of $\hat{\Delta}_n$, we say that it *removes* a letter of Q_n if that letter is not included in the subword of Q_n corresponding to F that is a reduced expression for w.

Let F be a facet of $\hat{\Delta}_{n+1}$. Define $\varphi(F)$ as follows:
In the first block of Q_{n+1}, consider the largest letter (not $2(n{+}1)$) that is removed by F. Since this must be even, we can label it as $2i$ for some $1 \le i \le n$. If the only letter removed in the first block is $2(n+1)$, we set $i = 0$.
Now, $\varphi(F)$ is the pair of facets F' in $\hat{\Delta}_i$ and F'' in $\hat{\Delta}_{n-i}$ given by the following procedure:

(1) Left-align the blocks of Q_i with the first i blocks of Q_{n+1}.
(2) For each letter removed by F in the first i blocks of Q_{n+1}, remove the corresponding letter in Q_i if one exists.
(3) Let F' be the set of indices that removes those letters from Q_i.
(4) Left-align the blocks of Q_{n-i} with the last $n-i$ blocks of Q_{n+1}. (The blocks are the same length, so they will in fact line up exactly.)
(5) For each letter removed by F in the last $n-i$ blocks, remove the corresponding letter in Q_{n-i}.
(6) Let F'' be the set of indices that removes those letters from Q_{n-i}.

We define $\varphi^{-1}(F', F'')$ to be the natural inverse of this map.

To illustrate the definitions of φ, φ^{-1}, consider the following examples.

Example 3.8. Consider the facet $\{4, 10, 12, 26, 28\}$ in $\hat{\Delta}_5$. We can write this as follows, where underlines denote the removal of a letter:

$$1\ 2\ 3\ \underline{4}\ 5\ 6\ 7\ 8\ 9\ \underline{10}\ 2\ \underline{3}\ 4\ 5\ 6\ 7\ 8\ 9\ 3\ 4\ 5\ 6\ 7\ 8\ 4\ \underline{5}\ 6\ \underline{7}\ 5\ 6$$

Here, we choose $i = 2$, so F' and F'' will each be facets of $\hat{\Delta}_2$. Aligning the blocks of Q_2 with the first 2 and last 2 blocks of Q_5 gives:

1	2	3	$\underline{4}$	5	6	7	8	9	$\underline{10}$	2	$\underline{3}$	4	5	6	7	8	9	3	4	5	6	7	8	4	$\underline{5}$	6	$\underline{7}$	5	6
1	2	3	4							2	3													1	2	3	4	1	2

Finally, we remove the corresponding letters to get our new facets:

$$\begin{array}{cccccccccccccccccccccccccccccc}
1 & 2 & 3 & \underline{4} & 5 & 6 & 7 & 8 & 9 & \underline{10} & 2 & \underline{3} & 4 & 5 & 6 & 7 & 8 & 9 & 3 & 4 & 5 & 6 & 7 & 8 & 4 & \underline{5} & 6 & \underline{7} & 5 & 6 \\
1 & 2 & 3 & \underline{4} & & & & & & & 2 & \underline{3} & & & & & & & & & & & & & 1 & \underline{2} & 3 & \underline{4} & 1 & 2
\end{array}$$

So $\varphi(\{4,10,12,26,28\}) = (\{4,6\},\{2,4\})$. Observe that each of these are, in fact, facets of $\hat{\Delta}_2$.

Now, consider the pair of facets $(\{2\},\{4,6,8\})$ in $\hat{\Delta}_1$ and $\hat{\Delta}_3$, respectively. We will use these to construct a facet in $\hat{\Delta}_5$. First, we left-align the block of Q_1 with the first block of Q_5. Then, we align the blocks of Q_3 with the last 3 blocks of Q_5. This gives us:

$$\begin{array}{cccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 3 & 4 & 5 & 6 & 7 & 8 & 4 & 5 & 6 & 7 & 5 & 6 \\
1 & 2 & & & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 2 & 3 & 4 & 5 & 3 & 4
\end{array}$$

Now, we remove the letters in Q_1 and Q_3 corresponding to our two facets:

$$\begin{array}{cccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 3 & 4 & 5 & 6 & 7 & 8 & 4 & 5 & 6 & 7 & 5 & 6 \\
1 & \underline{2} & & & & & & & & & & & & & & & & & 1 & 2 & 3 & \underline{4} & 5 & \underline{6} & 2 & \underline{3} & 4 & 5 & 3 & 4
\end{array}$$

Finally, we remove the corresponding letters in Q_5 and remove 10 to get our new facet.

$$\begin{array}{cccccccccccccccccccccccccccccc}
1 & \underline{2} & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \underline{10} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 3 & 4 & 5 & \underline{6} & 7 & \underline{8} & 4 & \underline{5} & 6 & 7 & 5 & 6 \\
1 & \underline{2} & & & & & & & & & & & & & & & & & 1 & 2 & 3 & \underline{4} & 5 & \underline{6} & 2 & \underline{3} & 4 & 5 & 3 & 4
\end{array}$$

So $\varphi^{-1}(\{2\},\{4,6,8\}) = \{2,10,22,24,26\}$. Observe that this is a facet of $\hat{\Delta}_5$.

Now, the proof of Theorem 3.7 can be completed by showing that φ, φ^{-1} are well-defined and always produce facets.

Proof. We will now show that φ is well defined and gives facets in $\hat{\Delta}_i$ and $\hat{\Delta}_{n-i}$. The choice of i is clearly well defined, so we need only show that the image is indeed a pair of facets.

Suppose $Q_n \setminus F = 1\ 2 \ldots \underline{2i} \ldots \underline{2n}\ 2\ 3 \ldots 2n{-}1 \ldots n\ n{+}1$, with underlined entries omitted. Then, using the Coxeter relations, this will evaluate to the same result as

$$1\,2 \ldots \underline{2i}\,2\,3 \ldots 2i{-}1 \ldots i{+}1 \,||\, 2i{+}1 \ldots 2n{-}1\ \underline{2n} \,|\, 2i \ldots 2n{-}1 \,|\, 2i{-}1 \ldots 2n{-}2 \,|\, \ldots \,|\, i{+}1 \ldots 2n{-}i \,|$$
$$|\ i{+}2 \ldots 2n{-}i{-}1 \ldots n\ n{+}1$$

We claim that before the line break, all omitted elements must occur prior to the double line, that the evaluation up to the double line is $(1\ 2i)(2\ 2i{-}1)\ldots(i\ i{+}1)$ and that the evaluation up to the line break is $(1\ 2n)(2\ 2n{-}1)\ldots(i\ 2n{-}i{+}1)(i{+}1\ 2n{-}i)$. This will imply that the word after the line break must evaluate to $(i{+}2\ 2n{-}i{-}1)\ldots(n\ n{+}1)$. If all these hold true, then our map is well defined, giving a facet in both $\hat{\Delta}_i$ and $\hat{\Delta}_{n-i}$.

We start by showing that the evaluation of this word up to the double line is $(1\ 2i)(2\ 2i{-}1)$ $\ldots(i\ i{+}1)$. By the definition of the fish product, it must be a product of disjoint transpositions and these transpositions must be contained in S_{2i}. Suppose the evaluation contains (k 2i). The rest of the first block evaluates to $(2i{+}1\ 2n)$. Evaluating the second block, our expression will then contain $(k\ 2n)$. If $k \neq 1$, we either had $(1\ m), m < i$, in which case 1 is now fixed, or $m \geq i$, in which case we get $(1\ 2n{-}2i{+}m)$. But we must have $(1\ 2n)$ as F is a facet. Hence, $k = 1$. Similarly, we get that the evaluation must contain $(2\ 2i{-}1)(3\ 2i{-}2)\ldots(i\ i{+}1)$. It must also be reduced, as F is a facet. Hence it contains i omitted elements and φ gives a facet in $\hat{\Delta}_i$

Then, in order to get $(1\ 2n)$ in our final expression, we must have all remaining elements of the first two blocks. This will then give us $(2\ 2i{-}1)$, which means we need all remaining elements of the next block in order to obtain $(2\ 2n{-}1)$. Similarly, we cannot omit any of the remaining elements of the first $i+1$ blocks. Furthermore, the evaluation of the first $i+1$ blocks is $(1\ 2i)(2\ 2i{-}1)\dots(i\ i{+}1)$ and we have omitted $i+1$ elements.

Hence, we must omit $n-i$ elements in the remaining blocks and it must evaluate to $(i{+}1\ 2n{-}i)\dots(n\ n{+}1)$, implying that φ gives a facet in $\hat{\Delta}_{n-i}$. Hence φ is well defined.

It is clear that φ^{-1} is well-defined, so we show that φ^{-1} produces a facet in $\hat{\Delta}_n$. Let F' be a facet in $\hat{\Delta}_i$ for some $0 \leq i \leq n-1$. Let F'' be a facet in $\hat{\Delta}_{n-i-1}$. Let $F = \varphi^{-1}(F', F'')$. We need to show that $Q_n \setminus F$ is a reduced expression for w_0.

Since F has i elements from F', $n-i-1$ elements from F'' and the additional element $2n$, it will have n elements. Hence, $Q_n \setminus F$ has the appropriate number of letters for a reduced expression for w_0. So, we just need to show that taking the ordered fish product of the elements of $Q_n \setminus F$ results in w_0.

As above, we can use the Coxeter relations to rewrite Q_n as

$$1\ 2\dots 2i\ 2\ 3\dots 2i{-}1\dots i\ i+1 \mid 2i{+}1\dots 2n\ 2i\dots 2n{-}1\ 2i{-}1\dots 2n{-}2\dots i{+}2\dots 2n{-}i{+}1$$

$$i{+}1\dots 2n{-}i \mid i{+}2\dots 2n{-}i{-}1\dots n\ n{+}1$$

By the definition of φ^{-1}, the word before the first vertical line has removals that correspond precisely to a facet of $\hat{\Delta}_i$. The word after the second vertical line has removals that correspond to a facet of $\hat{\Delta}_{n-i-1}$ with all letters incremented by $i+1$. The only removal between the two vertical lines is the removal of $2n$.

Since the portion before the first vertical line is a facet of $\hat{\Delta}_i$, the ordered fish product up to that point is $(1\ 2i)(2\ 2i{-}1)\dots(i\ i{+}1)$. The first block of the middle section has letters $2i{+}1\dots 2n{-}1$ since $2n$ is removed. None of the corresponding transpositions interact with $(1\ 2i)(2\ 2i{-}1)\dots(i\ i{+}1)$, so we can take their ordered product as if the first portion of the word was not there. This gives the transposition $(2i{+}1\ 2n)$. So, the result so far is $(1\ 2i)(2\ 2i{-}1)\dots(i\ i{+}1)(2i{+}1\ 2n)$

The second block of the middle section has letters $2i\dots 2n{-}1$. Taking the ordered fish product of that block with our result so far, we get $(2\ 2i{-}1)\dots(i\ i{+}1)(2i\ 2n{-}1)(1\ 2n)$. Notice that the transposition $(1\ 2n)$ will be fixed for the remainder of the product and that the remaining terms are analogous to the result before taking the fish product with the second block, only indexed to work with the third block in the middle section. By a similar argument, taking the ordered product of all remaining blocks in the middle section results in $(1\ 2n)(2\ 2n{-}1)\dots(i\ 2n{-}i{+}1)(i{+}1\ 2n{-}i)$.

Now, we know that the portion of the word after the second vertical line is a facet of $\hat{\Delta}_{n-i-1}$ with all letters incremented by $i+1$. Since the letters involved after the second vertical line are not found in our product so far, we can take their ordered product as if the rest of the word was not there. This final portion has the ordered fish product $(i{+}2\ 2n{-}i{-}1)(i{+}3\ 2n{-}i{-}2)\dots(n\ n{+}1)$.

Hence, our final result is $(1\ 2n)\dots(n\ n{+}1) = w_0$ and thus, F is a facet of $\hat{\Delta}_n$.

□

Corollary 3.9. *$F_n = C_n$, where C_n are the Catalan numbers.*

Theorem 3.10. *$\hat{\Delta}'_n$ and $\hat{\Delta}_n$ have the same number of facets.*

As with the previous theorem, in order to show this, we will establish a bijection between the facets of $\hat{\Delta}'_n$ and the facets of $\hat{\Delta}_n$.
Let F be a facet of $\hat{\Delta}_n$. Define $\psi(F)$ to be the facet F' in $\hat{\Delta}'_n$ given by the following procedure:

(1) Left-align all but the last letter of each block of Q'_n with the blocks of Q_n.
(2) Right-align the last letter of each block of Q'_n with the blocks of Q_n.
(3) For each letter removed by F in Q_n, remove the corresponding letter in Q'_n
(4) Let F' be the set of indices that removes these letters from Q'_n.

We define $\psi^{-1}(F')$ to be the natural inverse of this map.

Example 3.11. Consider the facet $\{8, 10, 16, 20, 22\}$ in $\hat{\Delta}_5$. To find $\psi(\{8, 10, 16, 20, 22\})$, we write Q_5 and $Q_5{}'$ as follows:

$$\begin{array}{cccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 3 & 4 & 5 & 6 & 7 & 8 & 4 & 5 & 6 & 7 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & 9 & 2 & 3 & 4 & 5 & 6 & 7 & & 8 & 3 & 4 & 5 & 6 & & 7 & 4 & 5 & & 6 & & 5
\end{array}$$

Then, we mark the letters removed by the facet $\{8, 10, 16, 20, 22\}$ in Q_5 and the corresponding letters in $Q_5{}'$.

$$\begin{array}{cccccccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \underline{8} & 9 & \underline{10} & 2 & 3 & 4 & 5 & 6 & \underline{7} & 8 & 9 & 3 & \underline{4} & 5 & \underline{6} & 7 & 8 & 4 & 5 & 6 & 7 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \underline{8} & & \underline{9} & 2 & 3 & 4 & 5 & 6 & \underline{7} & & 8 & 3 & \underline{4} & 5 & \underline{6} & & 7 & 4 & 5 & & 6 & & 5
\end{array}$$

So $\psi(\{8, 10, 16, 20, 22\}) = \{8, 9, 15, 18, 20\}$, which is a facet of $\hat{\Delta}'_5$.

For the proof of Theorem 3.10, it now suffices to show that ψ, ψ^{-1} are well-defined and always produce facets.

Proof. Let F be a facet in $\hat{\Delta}_n$. Let F' be the index set in $Q_n{}'$ given by $\psi(F)$. We align the words Q_n and $Q_n{}'$ as in the definition of ψ:

$$\begin{array}{ccccccccccccc}
1 & 2 & \cdots & 2n{-}1 & 2n & 2 & 3 & \cdots & 2n{-}2 & 2n{-}1 & \cdots & n & n{+}1 \\
1 & 2 & \cdots & & 2n{-}1 & 2 & 3 & \cdots & & 2n{-}2 & \cdots & & n
\end{array}$$

We need to show that F' is a facet of $\hat{\Delta}'_n$. Notice that it has the appropriate number of letters to make $Q_n{}' \setminus F'$ a reduced expression for $w_0{}'$. So, we only need to check that it evaluates to $w_0{}'$.

Choose i for F as in the definition of φ for Theorem 3.7. As in the proof of Theorem 3.7, the following sequences with appropriate removals will evaluate to the same results as $Q_n \setminus F$ and $Q_n{}' \setminus F'$ by the Coxeter relations:

$$\begin{array}{cccccccccccccccc}
1 & 2 & \cdots & 2i & 2 & 3 & \cdots & 2i{-}1 & \cdots & i & i{+}1 & | & 2i{+}1 & \cdots & 2n{-}1 & 2n \\
1 & 2 & \cdots & 2i & 2 & 3 & \cdots & 2i{-}1 & \cdots & i & i{+}1 & | & 2i{+}1 & \cdots & & 2n{-}1
\end{array}$$

$$\begin{array}{ccccccccccccc}
2i & \cdots & 2n{-}2 & 2n{-}1 & \cdots & i{+}2 & \cdots & 2n{-}i & 2n{-}i{+}1 & i{+}1 & \cdots & 2n{-}i{-}1 & 2n{-}i & | \\
2i & \cdots & & 2n{-}2 & \cdots & i{+}2 & \cdots & & 2n{-}i & i{+}1 & \cdots & & 2n{-}i{-}1 & |
\end{array}$$

$$\begin{array}{cccccccccc}
i{+}2 & \cdots & 2n{-}i{-}2 & 2n{-}i{-}1 & i{+}3 & \cdots & 2n{-}i{-}3 & 2n{-}i{-}2 & \cdots & n & n{+}1 \\
i{+}2 & \cdots & & 2n{-}i{-}2 & i{+}3 & \cdots & & 2n{-}i{-}3 & \cdots & & n
\end{array}$$

In the proof of Theorem 3.7, we argued that all of the removals in Q_n occur either before the first vertical line or after the second vertical line, except for $2n$. We also argued that the removals up to the first vertical line give a facet in $\hat{\Delta}_i$ and thus evaluate to $(1\ 2i)\dots(i\ i{+}1)$.

Since the letters of $Q_n{}'$ up to the first vertical line match exactly with the letters in Q_n, the removals given by F' will be the same as those given by F. So, the evaluation of $Q'_n \setminus F'$ up to the vertical line will be $(1\ 2i)\dots(i\ i{+}1)$ as well.

Between the two vertical lines, only $2n, \dots, 2n{-}i$ are removed. So, we take the ordered product with all of the letters in the middle section to get $(1\ 2n{-}1)(2\ 2n{-}2)\dots(i{+}1\ 2n{-}i{-}1)$.

Since the letters after the second vertical line don't interact at all with the result so far, we only need to verify that they evaluate to $(i{+}2\ 2n{-}i{-}2)\dots(n{-}1\ n{+}1)$. We will show this by induction. We want the result of this section to be exactly $w_0{}'$ for the $n-i-1$ case, but with all letters incremented by $i+1$. After the second vertical line, we have Q_n and $Q_n{}'$ aligned exactly as in the $n-i-1$ case, but with all letters incremented by $i+1$. Notice that we do have $n-i-1$ removals given by F and F' after the second vertical line since i removals occured before the first vertical line and $2n$ was removed in the middle section. Also, in the proof of Theorem 3.7, we argued that F gives a facet of $\hat{\Delta}_{n-i-1}$ in this section. So, we can repeat this process, breaking this section into three smaller sections as before. The first two sections will produce the product described above and the third will be shorter. It is easy to verify that this bijection holds for $n = 1$, so by induction the letters after the second vertical line evaluate to $(i{+}2\ 2n{-}i{-}2)\dots(n{-}1\ n{+}1)$.

Thus, $Q_n{}' \setminus F'$ evaluates to $w_0{}'$ and F' is a facet of $\hat{\Delta}_n{}'$. So ψ is well-defined.

The fact that ψ^{-1} is well-defined follows easily from a similar argument to that given for Theorem 3.7.

□

Remark 3.12. $\hat{\Delta}'_n$ and $\hat{\Delta}_n$ are isomorphic simplicial complexes.

Corollary 3.13. *$\hat{\Delta}'_n$ has C_n facets.*

Lemma 3.14. *$\hat{\Delta}_n{}'$ and $\hat{\Delta}_n$ each have $\binom{n+1}{2}$ vertices.*

Proof. From Remark 3.12, it suffices to proof this for $\hat{\Delta}_n$. We note that, by Corollary 3.4, we have at most $\binom{n+1}{2}$ vertices. To show that we have exactly this many, we proceed by induction. It is clear that this holds for $n = 1$. Suppose it is true for n. We can construct a facet of $\hat{\Delta}_{n+1}$ using a facet of $\hat{\Delta}_n$ in the $i = 0$ case of the above construction. Any facet of $\hat{\Delta}_n$ with the addition of $2(n+1)$ gives a facet of $\hat{\Delta}_{n+1}$ and hence we have $\binom{n+1}{2}$ vertices with index greater than $2(n{+}1)$. By lemma 3.5, so we have that $2, 4, \dots, 2(n{+}1)$ are also vertices, and thus we have at least $\binom{n+1}{2} + (n+1) = \binom{n+2}{2}$ vertices. Hence, we have exactly $\binom{n+2}{2}$ vertices, completing the proof. □

3.3. **Structure of Catalan Subword Complexes.** As with many Catalan objects, we can understand $\hat{\Delta}_n$ and $\hat{\Delta}_n{}'$ as the associahedron.

Theorem 3.15. *$\hat{\Delta}_n$ and $\hat{\Delta}_n{}'$ are the $(n{-}1)$-dimensional dual associahedron.*

Proof. As before, it suffices to show this for $\hat{\Delta}_n$. As in [8],we can characterize the $(n{-}1)$-dimensional dual associahedron as follows:

- The vertex set consists of internal diagonals in an $(n{+}2)$-gon.
- The face set consists of sets of mutually non-crossing diagonals.

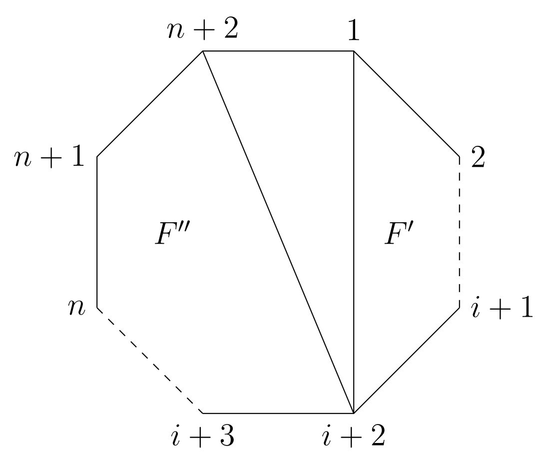
Hence, if we can establish a correspondence between our vertices and the internal diagonals, and our facets and triangulations, we are done.

We label the j-th even-indexed letter in the i-th block of Q_n with the diagonal $(i, j{+}i{+}1)$. This labeling can be given recursively, but it is clearer to state explicitly.

We now show that the facets of $\hat{\Delta}_n$ are precisely the triangulations of an $(n{+}2)$-gon, labeling the vertices as described above.

First, we show that any facet in $\hat{\Delta}_n$ corresponds to a triangulation. We proceed by induction.

Let $F \in \hat{\Delta}_n$ be a facet. As in the proof of the Catalan recurrence, we can break F into a facet $F' \in \hat{\Delta}_i$ and $F'' \in \hat{\Delta}_{n-i-1}$. Now, the $2i$ vertex in F' corresponds to the diagonal $(1, i{+}2)$ and the $2(n{-}i{-}1)$ vertex in F'' corresponds to the diagonal $(i{+}2, n{+}2)$. Now by the proof of Theorem 3.7, the facet F' has vertices in the first i blocks of Q_n where the position in block k is at most $2(i{-}k{+}1)$. Similarly, the facet F'' has vertices in the last $n{-}i{-}1$ blocks of Q_n. So, the facets F' and F'' correspond to diagonals completely contained in the $(i{+}2)$-gon and the $(n{-}i{+}1)$-gon shown below.



By induction, F' and F'' each correspond to triangulations, so F must be a triangulation of the $(n{+}2)$-gon.

Now, we show that any triangulation of an $(n{+}2)$-gon gives a facet in $\hat{\Delta}_n$. We proceed again by induction. Let T be a triangulation. It must have a triangle containing the edge $(1, n{+}2)$. Let the third vertex of that triangle be $i{+}2$. As before, split the $(n{+}2)$-gon into an $(i{+}2)$-gon with triangulation T' and a $(n-i-1)$-gon with triangulation T''. By induction, T' corresponds to a facet in $\hat{\Delta}_i$ and T'' corresponds to a facet in $\hat{\Delta}_{n-i-1}$. By the proof of Theorem 3.7, we can build a facet in $\hat{\Delta}_n$ corresponding to T.

These correspondences are clearly inverses. Thus, our complex and the $(n{-}1)$-dimensional dual associahedron have the same facets and are therefore the same polyhedron. $\square$

Definition 3.16. For two facets F_1 and F_2 of $\hat{\Delta}_n$, we say that F_1 *is related to* F_2 *by a right flip* if there exists a vertex $k \in F_1$ such that $F_2 = (F_1 \setminus \{k\}) \cup \{\ell\}$ for some $\ell > k$. That is, we can drop a vertex from F_1 and get a new facet F_2 of $\hat{\Delta}_n$ by choosing a new vertex from later in the word Q_n.

Definition 3.17. The *flip graph* of $\hat{\Delta}_n$ is the directed graph whose vertices are the facets of the complex $\hat{\Delta}_n$ and whose directed edges are those pairs (F_1, F_2) such that F_1 is related to F_2 by a right flip.

It will be convenient to discuss the flip graph in terms of the interpretation of the facets of $\hat{\Delta}_n$ as triangulations of an $(n+2)$-gon. In this setting, a triangulation T_1 is related to a triangulation T_2 by a right flip if there exists a diagonal $(i,j) \in T_1$ such that
$T_2 = (T_1 \setminus \{(i,j)\}) \cup (k,\ell)$ and $i < k, j < \ell$. We will always use the convention that the smaller vertex of the diagonal is listed first. Using our labeling of the diagonals of an $(n+2)$-gon with the vertices of $\hat{\Delta}_n$, it is easy to see that this definition and the definition given above will give the same directed graph.

It is known that the undirected version of this graph gives the vertices and edges of the associahedron. The directed version of this graph is often known as the Tamari lattice. See [12],[3] for more information. We will state certain well-known facts about the Tamari lattice which we can interpret as facts about $\hat{\Delta}_n$ and $\hat{\Delta}'_n$.

In order to better illustrate this graph, we will introduce the following biword representation of facets

Definition 3.18. The biword

$$\begin{matrix} i_1 & i_2 & \dots & i_{n-1} & i_n \\ j_1 & j_2 & \dots & j_{n-1} & j_n \end{matrix}$$

refers to the facet of $\hat{\Delta}(Q,w)_n$ obtained by omitting the $2j_k^{th}$ letter of the i_k^{th} block for each $1 \le k \le n$.

Definition 3.19. We will call the diagonal that (i,j) switches with to produce a new triangulation (regardless of whether or not it is a right flip) the *alternate* diagonal. Call a diagonal of a triangulation *stable* if it cannot be obtained from its alternate by a right flip.

Similarly, call a diagonal *unstable* if it can be obtained by a right flip of its alternate.

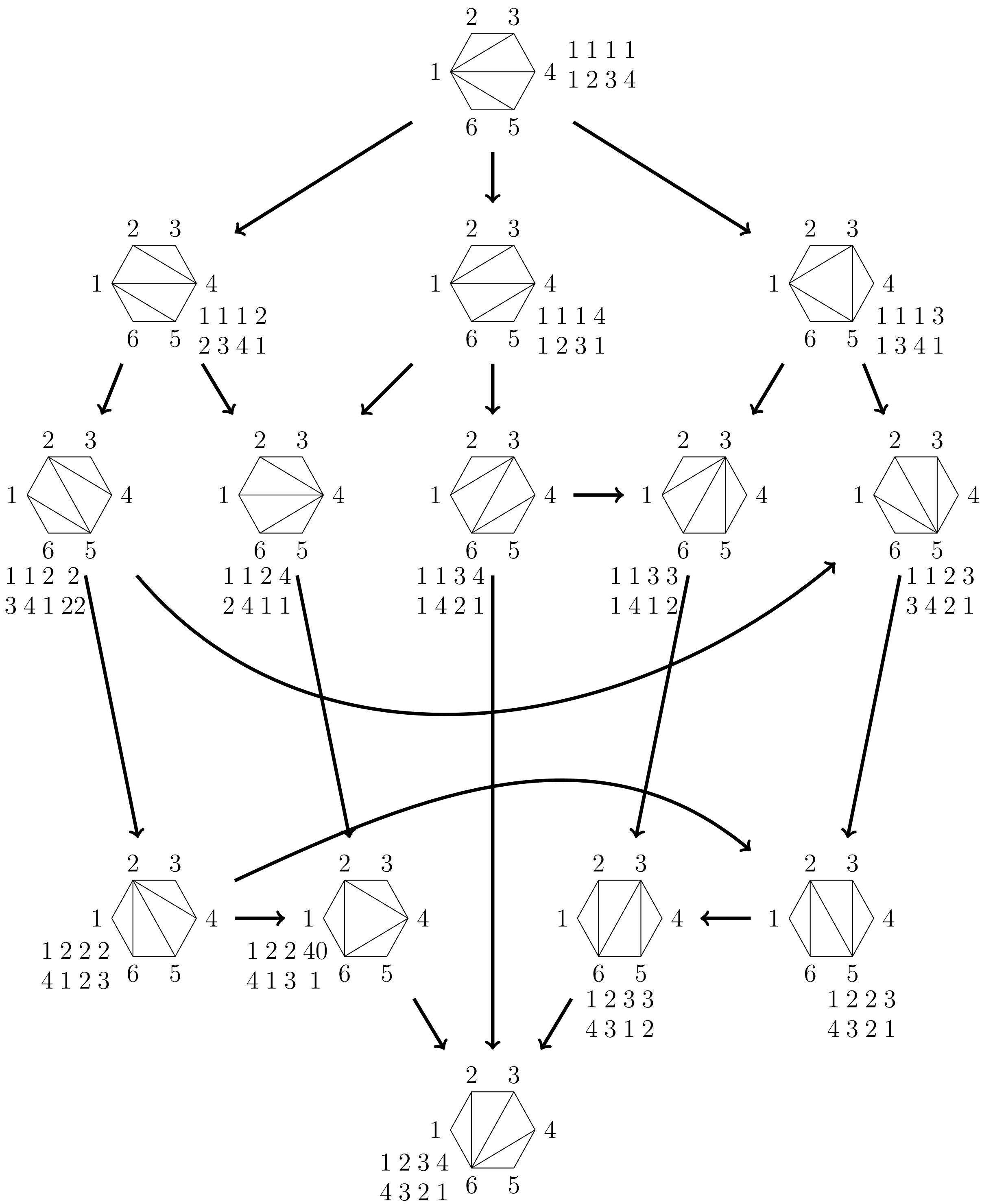
Call a triangulation of an n-gon *completely stable* if all of its diagonals are stable. That is, it is related to no other triangulations by a right flip. This will be a sink vertex in the flip graph. Similarly a *completely unstable* triangulation has all unstable diagonals and will be a source vertex.

Having established this correspondence, we state the following well-known results without proof.

Theorem 3.20. *The flip graph of $\hat{\Delta}_n$ has a unique source vertex, corresponding to a completely unstable triangulation, and a unique sink vertex, corresponding to a completely stable triangulation.*

Theorem 3.21. *For any triangulation T, there is a directed path in the flip graph from the source to T and from T to the sink.*

FIGURE 1. The flip graph of $\hat{\Delta}_4$ given by triangulations of the hexagon



4. K-Complexes

The following generalization of the discussed complex promises to be a fruitful area of study. Consider the complex $\hat{\Delta}(Q, w)$ for $w = 2n\ 2n{-}1\ \dots\ 1$ and

$$Q = Q_{n,k} = 1\ 2 \dots 2n\ 1\ 2 \dots 2n \dots 1\ 2 \dots 2n\ 2 \dots 2n{-}1\ 3 \dots 2n{-}2 \dots n\ n{+}1,$$

where the sequence $1\ 2 \dots 2n$ is repeated k times. The previous section discussed the $k = 1$ case. We now consider this complex for general k.

Lemma 4.1. *The vertex set of* $\hat{\Delta}_{n,k} = \hat{\Delta}(Q_{n,k}, 2n \dots 1)$ *is* $\{1, 2, \dots n(n+2k-1)\}$ *for* $k \geq 2$.

Proof. Note that we can omit any increasing block and $Q_{n,k}$ will still have as a subword a copy of $Q_{n,1}$ and hence a reduced word for $2n\ 2n{-}1 \dots 1$. The result follows. □

Lemma 4.2. $\hat{\Delta}_{n,k}$ *has at least* $\binom{n+k-1}{k-1} \cdot C_n$ *facets.*

Proof. For each facet of $\hat{\Delta}_{n,1}$, we can choose any $k-1$ of the increasing blocks to completely omit and use the facet of $\hat{\Delta}_{n,1}$ to get a reduced word in the remaining blocks (which contain $Q_{n,1}$ as a subword). The result follows from the observation that each choice of facet and blocks will give a distinct facet in $\hat{\Delta}_{n,k}$. □

This complex is analogous to one discussed in [2]. We hope to further explore this complex and its properties in the future.

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