A Study of Sequences involving Factorials

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Abstract

In this paper, we aim to study sequences which are products of factorials and examine the conditions required for them to be integral. Using the order test, we found necessary and sufficient conditions for these sequences to be integral and our study of these conditions led us to study cyclotomic polynomials, providing more efficient methods of studying these sequences.
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1 Integral Ratios of Factorials

Consider two vectors $\mathbf{a} \in \mathbf{N^k}$, $\mathbf{b} \in \mathbf{N^l}$, with $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_l)$. Then define a sequence of rational numbers

$$t_n = \frac{(a_1 n)! \ldots (a_k n)!}{(b_1 n)! \ldots (b_l n)!} = \frac{\prod_{i=1}^{k} (a_i n)!}{\prod_{j=1}^{l} (b_j n)!}$$

where n is any positive integer. The subject of this paper is to investigate the conditions for $\mathbf{a}$ and $\mathbf{b}$ to be such that the sequence $\{t_n\}$ is integral, i.e $t_n \in \mathbf{Z}$ for all $n \geq 1$

There are many examples of sequences of this type which will always be integral, such as

$$t_n = \frac{(2n)!}{(n)!(n)!} = \binom{2n}{n}$$

or

$$t_n = \frac{(30n)!}{(15n)!(10n)!(5n)!} = \binom{30n}{15n\ 10n\ 5n} = \binom{30n}{5n}\binom{25n}{10n}$$

can both be expressed as a product of binomial coefficients, and will therefore always take integer values for all n.
However, sequences such as $t_n = \frac{(6n)!n!}{(2n)!(2n)!(3n)!}$ cannot be expressed as a product of binomial coefficients. To see this, consider a product of m binomial coefficients. There will be m terms in the numerator and $2m$ in the denominator. If k of these can be cancelled, we are left with $m-k$ terms in the numerator and $2m-k$ in the denominator. In this case, we would have $m-k=2$ and $2m-k=3$, which has no solutions in non-negative integers. Hence $t_n = \frac{(6n)!n!}{(2n)!(2n)!(3n)!}$ cannot be the product of binomial coefficients. It can also be shown that it is always integral, as follows.

2 Criterion of Integrality

Definition: Let p be a prime and $n \in \mathbf{Z}, \mathbf{n} \neq \mathbf{0}$. The *order* of n, denoted by $ord_p(n)$, to a prime p is defined as the largest nonnegative integer m such that p^m divides n. For a rational number $x = \frac{y}{z}$, define $ord_p(x) := ord_p(y) - ord_p(z)$.

Clearly a rational number $x \in \mathbf{Q}$ is integral if and only if $ord_p(x) \geq 0$ for all primes p.

Proposition 2.1

$$ord_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$$

Proof There are $\lfloor \frac{n}{p} \rfloor$ integers below n that contribute a factor p. Of these, $\lfloor \frac{n}{p^2} \rfloor$ contribute a second factor; and among those $\lfloor \frac{n}{p^3} \rfloor$ contribute a third factor p, and so on. This will be a finite sum as all terms will be 0 for sufficiently large k ∎

Note that the order of a product of factorials is additive, so this gives the condition that for a sequence to always take integer values, the order of the numerator must be greater than or equal to the order of the denominator for all primes p and all values of n.
We next provide an alternative proof for a result by Edmund Landau

Definition: Given a sequence $t_n = \frac{\prod_{i=1}^{k}(a_i n)!}{\prod_{j=1}^{l}(b_j n)!}$, define the Landau function f, $f : [0,1] \to \mathbf{Z}$ such that $f(x) = \sum_{i=1}^{k} \lfloor a_i x \rfloor - \sum_{j=1}^{l} \lfloor b_x \rfloor$.

This function seems to apper for the first time in relation to ratios of factorials in the paper *Sur les conditions de divisibilité d'un produit de factorielles par un autre, Edmund Landau*

Theorem 2.2 (Function Test) *A sequence $\{t_n\}$ is integral if and only if, $\forall x \in [0,1], f(x) \geq 0$, where $f(x)$ is the Landau function associated with the sequence.*

Proof First suppose $f(x) \geq 0 \; \forall x \in [0,1]$. We will show this implies that t_n is integral.
Consider the expression

$$\zeta = \sum_i \lfloor \frac{a_i n}{p^k} \rfloor - \sum_j \lfloor \frac{b_j n}{p^k} \rfloor$$

for $n, k \in \mathbf{N}$ and p a prime. Let $y_k = \frac{n}{p^k}$ and $x_k = \{y_k\}$, so that

$$y_k = \lfloor y_k \rfloor + x_k, x_k \in [0,1)$$

$$Then \zeta = \sum_i \lfloor a_i \lfloor y_k \rfloor + a_i x_k \rfloor - \sum_j \lfloor b_j \lfloor y_k \rfloor + b_j x_k \rfloor$$

$$= \sum_i a_i \lfloor y_k \rfloor \; - \sum_j b_j \lfloor y_k \rfloor \; + \sum_i \lfloor a_i x_k \rfloor \; - \sum_j \lfloor b_j x_k \rfloor$$

$$= (\sum_i a_i \; - \sum_j b_j) \lfloor y_k \rfloor \; + \; f(x_k)$$

Since $\sum_i a_i - \sum_j b_j = f(1) \geq 0$ and $f(x) \geq 0, \forall x$, this implies $\zeta \geq 0$.

$$\Rightarrow \sum \lfloor \frac{a_i n}{p^k} \rfloor - \sum \lfloor \frac{b_i n}{p^k} \rfloor \geq 0 \; \forall n, k \in \mathbf{N}$$

where p is any prime. Therefore

$$ord_p(t_n) = \sum_{k=1}^{\infty} (\sum \lfloor \frac{a_i n}{p^k} \rfloor - \sum \lfloor \frac{b_i n}{p^k} \rfloor) \geq 0$$

for all n and p, which implies that the sequence $\{t_n\}$ is integral.
Conversely, suppose t_n is intergral. We will show this implies that $f(x) \geq 0 \; \forall x \in [0,1]$.
First define $L = lcm(a_1, ..., a_k, b_1, ..., b_l)$. Next suppose $f(x) < 0$ for some x, implying $f(x) < 0 \; \forall x \in (\frac{k}{L}, \frac{k+1}{L})$ for some k, as jumps in $f(x)$ can only occur at x such that $a_i x \in \mathbf{Z}$ or $b_j x \in \mathbf{Z}$, for some $iorj$, and that can only occur when $x = \frac{d}{a_i} = \frac{c}{L}$ or $x = \frac{d}{b_j} = \frac{c}{L}$. Next take $p > L$ to be rpime. Suppose there does not exist some m, $0 \leq m \leq p-1$ such that $\frac{m}{p} \in [\frac{k}{L}, \frac{k+1}{L}]$ This means $\frac{m}{p} \leq \frac{k}{L}$ and $\frac{k+1}{L} \leq \frac{m+1}{p}$. Therefore $mL < kp$ and similarly $kp + p < mL + L$.
$\Rightarrow mL < kp < mL + L - p$ But this implies $L - p > 0 \; \Rightarrow l > p$, a contradiction, so $\exists \; m$ such that $f(\frac{m}{p}) < 0$.
Note that, as $p^2 > Lp > a_{max} p > a_{max} m \geq a_i m$ and

$p^2 > Lp > b_{max}p > b_{max}m \geq b_j m$ we must have $\lfloor \frac{a_i m}{p^k} \rfloor = \lfloor \frac{b_j m}{p^k} \rfloor = 0 \; \forall k \geq 2$.

$$Henceord_p(t_m) = \sum_i \lfloor \frac{a_i m}{p} \rfloor - \sum_j \lfloor \frac{b_j m}{p} \rfloor = \sum_i \lfloor a_i \frac{m}{p} \rfloor - \sum_j \lfloor b_j \frac{m}{p} \rfloor = f(\frac{m}{p}) < 0$$

This implies $t_m \notin \mathbf{Z}$, implying t_n is not integral, a contradiction. Therefore we must have $f(x) \geq 0 \; \forall \; x$.
Therefor t_n is integral if and only if $\sum a_i \geq \sum b_i$ and $f(x) \geq 0 \; \forall \; x \in [0, 1]$. ∎

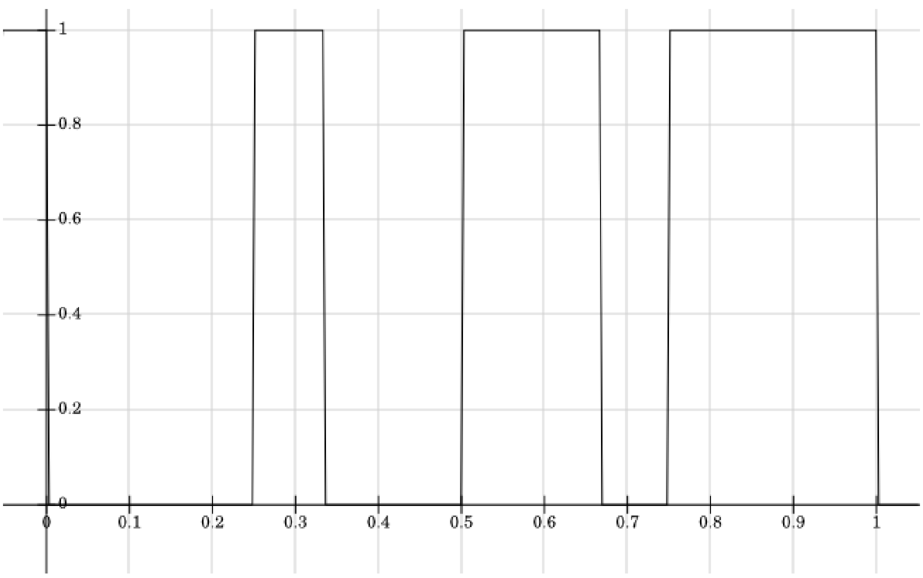
3 Some Examples

There are several families of sequences which will always produce integer values. Sequences which are simply products of binomial coefficients, as were discussed earlier are trivially integral. It can also be shown that sequences of the general form

$$t_n = \frac{(2an)!(2bn)!}{(an)!(bn)!((a+b)n)!}$$

will always take integer values, where a and b are any positive integers. This can also be generalised to sequences with more terms. We will now prove that the above sequence is always integral, where a and b are positive integers.

Proof The sequence is integral if and only if the associated Landau function is non-negative. Graphing $f(x) = \lfloor 2ax \rfloor + \lfloor 2bx \rfloor - \lfloor ax \rfloor - \lfloor bx \rfloor - \lfloor (a+b)x \rfloor$ we get



As this is non-negative, our sequence must be integral ∎

4 Jumps of the Landau Function

Graphing the Landau function, we notice that it is a step function and hence is piecewise constant and right continuous. So, to check if $f(x) \geq$, it is sufficient to check that $f(x) \geq 0$ at each of the jumps. These jumps occur when a term in $f(x)$ take a new integer value and so, it is in fact sufficient to check that $f(x) \geq 0$ for all $x = \frac{i}{L}$ where $L = lcm(a_1, a_2, ..., a_k, b_1, b_2, ..., b_l)$ and $i = \{0, 1, ..., L\}$. However, we do not get a jump at each of these points. In this section our goal is to determine precisely at which of the points $\frac{i}{L}$ these jumps occur. First, we need to introduce a family of polynomials known as cyclotomic polynomials.

Definition: A *Cyclotomic Polynomial*, $\Phi_n(x)$, where n is a positive integer, is the unique monic polynomial defined by the following formula:

$$\Phi_n(x) := \prod_{1 \leq k < n : gcd(k,n)=1} (x - e^{2\pi i \frac{k}{n}})$$

Cyclotomic polynomials have the following properties

Proposition 4.1
- *$\Phi_n(x)$ has integral coefficients*
- *$\Phi_n(x)$ is irreducible over $\mathbf{Q}$, that is, it cannot be written as a product of two polynomials with rational coefficients*
- $x^n - 1 = \prod_{d|n} (\Phi_d(x))$

Below are the first 7 cyclotomic polynomial:

- $\Phi_1(x) = x - 1$
- $\Phi_2(x) = x + 1$
- $\Phi_3(x) = x^2 + x + 1$
- $\Phi_4(x) = x^2 + 1$
- $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$
- $\Phi_6(x) = x^2 - x + 1$
- $\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

Next we define $a(x) = \prod_{i=1}^{k} (x^{a_i} - 1)$ and $b(x) = \prod_{j=1}^{l} (x^{b_j} - 1)$.

We then define $p(x)$ and $q(x)$ such that $\frac{p(x)}{q(x)} = \frac{a(x)}{b(x)}$, but $p(x)$ and $q(x)$ have no common factors. $p(x)$ and $q(x)$ can then be written as a product of cyclotomic polynomicals. These cyclotomic polynomials have roots at $e^{2\pi i(\alpha_i)}, 1 \leq i < s$, and $e^{2\pi i(\beta_j)}, 1 \leq j < t$, respectively, with $\alpha_i, \beta_j \in (0, 1]$ and $s \leq k$ and $t \leq l$.We call α_i and β_j the *argument* of the associated root. We will show that jumps upward occur exactly at α_i and jumps downward occur exactly at β_j, leading to an alternative definition of the Landau function.

Theorem 4.2 *Given a sequence $\{t_n\}$, we have the following identity for the associated Landau function.*

$$f(x) = \#\{i : \alpha_i \leq x\} - \#\{j : \beta_j \leq x\} x \in [0, 1]$$

where $\{\alpha_i\}, \{\beta_j\}$ are defined as above.

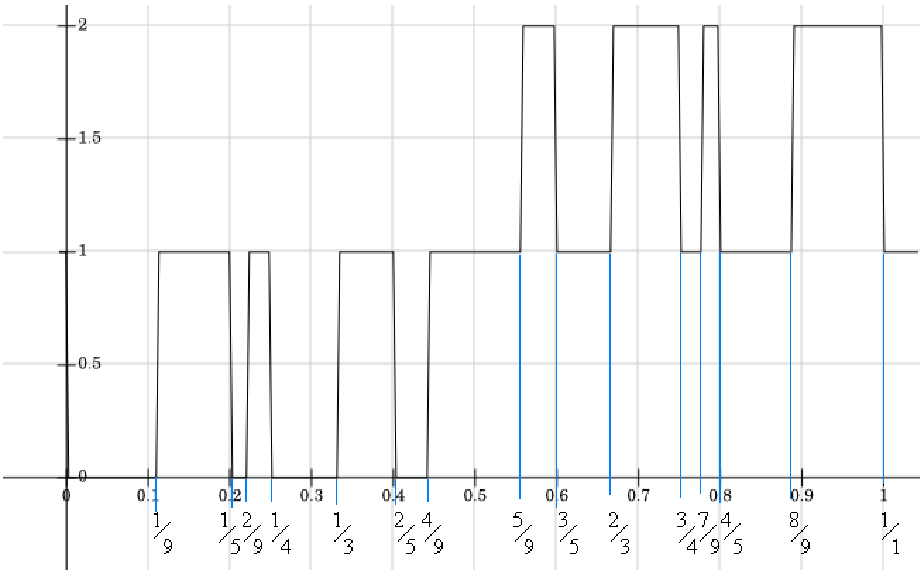
Consider, for example the sequence $t_n = \frac{(9n)!(2n)!}{(n)!(4n)!(5n)!}$. The corresponding functions are given by

$$a(x) = (x^9 - 1)(x^2 - 1) = \Phi_1(x)\Phi_3(x)\Phi_9(x)\Phi_1(x)\Phi_2(x)$$

and

$$b(x) = (x - 1)(x^4 - 1)(x^5 - 1) = \Phi_1(x)\Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_1(x)\Phi_5(x)$$

which gives $\frac{a(x)}{b(x)} = \frac{p(x)}{q(x)}$ implying $p(x) = \Phi_3(x)\Phi_9(x)$ and $q(x) = \Phi_1(x)\Phi_4(x)\Phi_5(x)$. This gives $\alpha_1 = \frac{1}{3},\ \alpha_2 = \frac{2}{3},\ \alpha_3 = \frac{1}{9},\ \alpha_4 = \frac{2}{9},\ \alpha_5 = \frac{4}{9},\ \alpha_6 = \frac{5}{9},\ \alpha_7 = \frac{7}{9},\ \alpha_8 = \frac{8}{9}$ while $\beta_1 = 1,\ \beta_2 = \frac{1}{4},\ \beta_3 = \frac{3}{4},\ \beta_4 = \frac{1}{5},\ \beta_5 = \frac{2}{5},\ \beta_6 = \frac{3}{5},\ \beta_7 = \frac{4}{5},$ which, as we can see corresponds to the jumps in the graph below.



This holds in general and we can prove this as follows.

Proof Note that $\lfloor m \rfloor = \#\{k \in \mathbf{N} : \mathbf{k} \leq \mathbf{m}\}$.
Hence $\lfloor mx \rfloor = \#\{k : k \leq mx\} = \#\{\frac{k}{m} : \frac{k}{m} \leq x\}$.
So $f(x) = \sum \lfloor a_i x \rfloor - \sum \lfloor b_i x \rfloor = \sum \#\{\frac{k}{a_i} : \frac{k}{a_i} \leq x\} - \sum \#\{\frac{k}{b_j} : \frac{k}{b_j} \leq x\}$

$= \#\{\text{Roots of } a \text{ with argument} \leq x\} - \#\{\text{Roots of } b \text{ with argument} \leq x\}$
$= \#\{\alpha_i \leq x\} - \#\{\beta_j \leq x\}$

∎

This allows us to quickly and easily find the graph corresponding to a sequence by hand.

5 Reconstructing a Sequence

Another question which arises from our study of these polynomials is if we can reconstruct a sequence given $p(x)$ and $q(x)$ as products of cyclotomic polynomial. We will now present an algorithm to do this:

Definition: Define:$max\{\phi_{a_1}, \phi_{a_2}, \phi_{a_3}, \ldots, \phi_{a_s}\} = \phi_{max\{a_1,a_2,\ldots,a_s\}}$

Suppose $p(x) = \prod_{i=1}^{s} \phi_{a_i}$ and $q(x) = \prod_{j=1}^{t} \phi_{b_j}$ Let A_0 be the set of ϕ_{a_i} and B_0 the set of ϕ_{b_j}. We then proceed with the following algorithm:

1. Let $\phi_{s_i} = max\{\phi_i : \phi_i \in A_i \cup B_i\}$
   If $\phi_{s_i} \in A_i$, let $a_{i+1} = s$
   If $\phi_{s_i} \in B_i$, let $b_{i+1} = s$

2. Let $C_i = \{\phi_d : d \mid s_i, 0 < d < s_i\}$
   If $\phi_{s_i} \in A_i$:, let $A_{i+1} = A_i \setminus \{\phi_s\}$. Let $B_{i+1} = B_i \cup C_i$.
   If $\phi_{s_i} \in B_i$:, let $A_{i+1} = A_i \cup C_i$. Let $B_{i+1} = B_i \setminus \{\phi_s\}$.

3. If $\mid A_{i+1} \mid + \mid B_{i+1} \mid > 0$, then repeat. Else, end.

Note that this algorithm will terminate, as s_i is decreasing and when $s_i = 1$, $|C_i| = 0$ and so $|A_i| + |B_i|$ is strictly decreasing and must eventually become 0. We then have two sequences $\{a_i\}$, $\{b_i\}$ and can cancel any common values, giving us $\mathbf{a}$, $\mathbf{b}$ and t_n. $\blacksquare$

Let us illustrate this algorithm with and example. For example,consider $p(x) = \phi_6$ and $q(x) = \phi_1\phi_2$ Then $A_0 = \{\phi_6\}$ and $B_0 = \{\phi_1, \phi_2\}$ and $\phi_{s_0} = \phi_6$. So we get $a_1 = 6$ and $C_0 = \{\phi_1, \phi_2\phi_3\}$
This gives $A_1 = \{\emptyset\}$ and $B_1 = \{\phi_1, \phi_1, \_phi_2, \phi_2, \phi_3\}$. So $\phi_{s_1} = \phi_3$ and we have $b_1 = 3$ and $C_1 = \{\phi_1\}$
This give $A_2 = \{\phi_1\}$ and $B_2 = \{\phi_1, \phi_1, \phi_2, \phi_2\}$. Repeating this process we find $a_1 = 6, a_2 = 1, a_3 = 1, a_4 = 1$ and $b_1 = 3, b_2 = 2, b_3 = 2, b_4 = 1, b_5 =_1$. Cancelling common terms, we get that the corresponding sequence is given be $t_n = \frac{(6n)!(n)!}{(3n)!(2n)!(2n)!}$
The existence of this algorithm further cements the relationship between our sequences and cyclotomic polynomials, implying there may be a condition on $p(x)$ and $q(x)$ to test if the corresponding sequence is integral.

6 Remarks

On completing this project we had determined some necessary and sufficient conditions for such sequences to be integral and have developed efficient methods for testing sequences, We created a new proof of Landau's function. as well as finding an alternative formulation of the Landau function from our study of cyclotomic polynomials. Were we to pursue this research further we would investigate the conditions on the associated products of cyclotomic polynomials for a sequence to be integral. We would also try and identify other families of integer sequences as were discussed previously.