Multiple zeta values and motivic periods

A lecture course for the University of Bonn

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Lecture 1

1.1 The algebra of periods

When first constructing the complex numbers, we normally start with the monoid of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

which we complete into the ring of integers $\mathbb{Z}$, and then take the field of fractions in order to obtain the rational numbers $\mathbb{Q}$. From there we have two routes to the complex numbers.

$$\begin{array}{ccc} \mathbb{Q} & \subset & \bar{\mathbb{Q}} \\ \cap & & \cap \\ \mathbb{R} & \subset & \mathbb{C} \end{array}$$

We may first take the topological completion to obtain the reals $\mathbb{R}$ and then the algebraic closure to obtain the complex numbers $\mathbb{C}$, or we may first take the algebraic closure to obtain the field of algebraic numbers

$$\bar{\mathbb{Q}} := \{z \in \mathbb{C} \mid \exists a_0, a_1, , \ldots, a_n \in \mathbb{Q} \text{ such that } a_n z^n + \cdots + a_0 = 0\}$$

and then take the topological completion to obtain $\mathbb{C}$ once again.A complex number which is not algebraic is called transcendental.

The set of algebraic numbers is much more structured than that of the reals. The set of algebraic numbers is countable, unlike the reals. The field $\mathbb{Q}(\alpha)$ is finite dimensional over $\mathbb{Q}$ for all algebraic α, but not all real α, and we have all the machinery coming from Galois theory to help us answer questions in algebraic numbers that we lack for more general complex numbers. However, many important mathematical constants (π, e, γ), are transcendental.

The set of periods provides an intermediate set between the algebraic numbers and the complex numbers. It is countable, but contains many transcendental numbers and other numbers of interest, such as π, $\zeta(2n+1)$, values of L-functions and hypergeometric functions (up to powers of π). Furthermore, the theory of motives offers us a notion of Galois theory for periods, giving us a significant amount of structure on an expansive class of numbers.

1.1.1 First definitions and examples

The following definitions are due to Kontsevich and Zagier [10], and will serve as our initial definitions of a period.

Definition 1.1.1. A (naive) period is a complex number whose real and imaginary parts are given by absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Example 1.1.2. The set of periods contains all rational and algebraic numbers

$$2 = \int_0^2 dx$$
$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

along with π and logarithms of rational numbers

$$\pi = \int_{x^2+y^1 \leq 1} dxdy$$
$$\ln 3 = \int_1^3 \frac{dx}{x}$$

and values of the zeta function

$$\zeta(3) = \int_{0 \leq x \leq y \leq z} \frac{dxdydz}{(1-x)yz}$$

We denote by $\mathcal{P}$ the set of periods. We also remark that we can replace rational functions having rational coefficients by algebraic functions having algebraic coefficients in our above definition, simply by introducing more variables as needed.

Example 1.1.3. Consider the integral

$$I = \int_0^{\sqrt{3}} \sqrt{2}\sqrt{1-x}dx.$$

By introducing new variables, we can rewrite this as an integral of a rational function over a rationally defined domain.

$$I = \int_{\substack{0 \leq x \\ x^2 \leq 3}} \int_{2y^2 \leq 1} \int_{\substack{0 \leq z \\ z^2 \leq 1-x}} dxdydz$$

While the above of periods is elementary and accessible, it will often be convenient to work with a seemingly more general, but equivalent, defintion.

Definition 1.1.4. Let X be a smooth, quasiprojective variety defined over $\bar{\mathbb{Q}}$,, Y a subvariety defined over $\bar{\mathbb{Q}}$, and ω an algebraic n-form on X that vanishes on Y (again defined over $\bar{Q}$). Let C a singular n-form on $X(\mathbb{C})$, with boundary contained in Y. Then the integral $\int_C \omega$ is a period.

Example 1.1.5. Taking $X = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, $Y = \emptyset$, $\omega = \frac{dz}{z}$, and C a circle around 0, we obtain

$$2\pi i = \int_{|z|=1} \frac{dz}{z}$$

as a period. Similarly, taking $X = \mathbb{G}_m$, $Y = \{1, 2\}$, $\omega = \frac{dz}{z}$ and C the straight line path from 1 to 2, we obtain

$$\ln 2 = \int_1^2 \frac{dz}{z}$$

as a period.

Unlike the algebraic numbers and complex numbers, periods do not form a field. They do, however form a ring, with an obvious $\mathbb{Q}$-algebra structure.

Theorem 1.1.6. *The set of periods $\mathcal{P}$ is a ring,*

Proof. Suppose we have two periods given by the data $(X_1, Y_1, \omega_1, C_1)$, and $X_2, Y_2, \omega_2, C_2)$. Then, letting $\pi_i : X_1 \times X_2 \to X_i$ be the standard projection, we can write the product

$$\left(\int_{C_1} \omega_1\right)\left(\int_{C_2} \omega_2\right) = \int_{C_1 \times C_2} \pi_1^* \omega_1 \pi_2^* \omega_2$$

as a integral on $X_1 \times X_2$, over a chain with boundary in $Y_1 \times Y_2$.

To show the sum is again a period, note that the integral corresponding to $(\mathbb{A}^1, \{0,1\}, dz, [0,1])$ gives period 1. We may multiply our periods by this in order to assume that X_1 and X_2 exist in ambient spaces of the same dimension. As such, we can form their disjoint union $X_1 \bigsqcup X_2$, on which we have the differential form $\omega_1 + \omega_2$ and chain $C_1 \bigsqcup C_2$ with boundary in $Y_1 \bigsqcup Y_2$. Then

$$\int_{C_1} \omega_1 + \int_{C_2} \omega_2 = \int_{C_1 \bigsqcup C_2} \omega_1 + \omega_2.$$

□

Remark 1.1.7. The ring $\mathcal{P}$ is often called the set of *effective* periods, and the ring of periods is defined to be $\mathcal{P}[\frac{1}{\pi}]$. This will be a useful convention to adopt when discussing motivic periods, and enables us to consider a wider range of numbers as periods, such as special values of L-functions, or values of the hypergeometric function

$${}_2F_1(a, b; c; x) := \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

which lie in $\frac{1}{\pi}\mathcal{P}$ for $a, b, c \in \mathbb{Q}$, $x \in \bar{\mathbb{Q}}$, $|x| \leq 1$. We may at some point use period to refer to both effective periods and elements of $\mathcal{P}[\frac{1}{\pi}]$, but it should be clear from context.

In general it is difficult to tell whether a number is a period or not. Some numbers can be shown to be periods, despite non having an obvious integral representation, such that the logarithmic Mahler measure

$$\mu(P) := \int_{|x_i|=1} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \text{ for } P \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

where $\mu(P) \in \mathcal{P}$ [7]. Similarly, while

$$\Gamma(s) := \int_0^\infty t^{s-1} e^- t dt$$

is in general not expected to be a period, for $t = \frac{p}{q} \in \mathbb{Q}$, $\Gamma(\frac{p}{q})^q \in \mathcal{P}$.

On the flip side, it is also quite difficult to show a number is not a period. While $e, \gamma, \frac{1}{\pi}$ are conjectured to be non-periods, there are no known natural examples of non-periods.

Remark 1.1.8. While there are no known natural examples of non-periods, Yoshinaga constructs an explicit example of a computable non-period [11] by using the fact that periods are somehow 'easily' computable.

While it is, in general, difficult to determine whether a given number is a period, it is conjecturally easy to check whether two periods are equal.

Conjecture 1.1.9 (Kontsevich-Zagier [10])**.** If two periods are equal, they can be related by a chain of equalities using

1. Linearity:

$$\int_C \omega_1 + \omega_2 = \int_C \omega_1 + \int_C \omega_2$$
$$\int_{C_1 \sqcup C_2} \omega = \int_{C_1} \omega + \int_{C_2} \omega$$

2. Change of variables: For any invertible $f : X_1 \to X_2$, ω an n-form on X_2, C an n-chain on X_1

$$\int_{f_*(C)} \omega = \int_C f^*(\omega)$$

3. Stokes' theorem:

$$\int_C d\omega = \int_{\delta C} \omega$$

Example 1.1.10. We can use these relations to prove $\zeta(2) := \sum_{n\geq 1} \frac{1}{n^2}$ is equal to $\frac{\pi^2}{6}$. Let

$$I := \int_0^1 \int_0^1 \frac{1}{1-xy} \frac{dxdy}{\sqrt{xy}}$$

By expanding this as a geometric series and integrating term by term, we find

$$\begin{aligned} I &= \sum_{n\geq 0} \int_0^1 \int_0^1 x^{n-1/2} y^{n-1/2} dxdy \\ &= \sum_{n\geq 0} \frac{1}{(n+1/2)^2} \\ &= 4 \sum_{n\geq 0} \frac{1}{(2n+1)^2} \\ &= 4 \left(\zeta(2) - \sum_{n>0} \frac{1}{(2n)^2} \right) \\ &= 4\zeta(2) - \zeta(2) = 3\zeta(2). \end{aligned}$$

However, if we make the change of variables

$$x = \xi^2 \frac{1+\eta^2}{1+\xi^2},\ y = \eta^2 \frac{1+xi^2}{1+\eta^2}$$

we can show

$$\begin{aligned} I &= 4\int_0^1\int_0^1 \frac{d\xi}{1+\xi^2}\frac{d\eta}{1+\eta^2} \\ &= 2\int_0^\infty\int_0^\infty \frac{d\xi}{1+\xi^2}\frac{d\eta}{1+\eta^2} \\ &= \frac{\pi^2}{2} \end{aligned}$$

Thus, $3\zeta(2) = \frac{\pi^2}{2}$ and the result follows.

1.2 Multiple zeta values

1.2.1 Values of the Riemann zeta function

Recall the definition of the Riemann zeta function $\zeta(s) := \sum_{0<n} \frac{1}{n^s}$. This series is defined for real part of s greater than 1. While it can be meromorhically continued to $\mathbb{C}$, we are more interested in its values at positive integers. The values at even positive integers are well understood.

Theorem 1.2.1 (Euler)**.** *Define the Bernoulli numbers by the series $\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x}{e^x-1}$. These are rational numbers and we have*

$$\zeta(2n) = \frac{-B_{2n}}{2(2n!}(2\pi i)^{2n}$$

for all $n > 0$.

Proof. There are a number of proof methods, using integrals, Fourier analysis, etc. We will prove this using generating series. Recall that

$$\frac{\sin \pi x}{\pi x} = \prod_{n>0}\left(1 - \frac{x^2}{n^2}\right).$$

Taking logarithmic derivatives, we find that

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n>0} \frac{1}{x+n} + \frac{1}{x-n}$$

Expanding the right hand side using geometric series, we find that

$$\pi i \frac{e^{\pi i x} + e^{-\pi i x}}{e^{\pi i x} - e^{-\pi i x}} = \frac{1}{x} - \sum_{n>0} 2\zeta(2n)x^{2n-1}$$

We can then rewrite the left hand side as

$$\begin{aligned} \pi i \frac{e^{2\pi i x}+1}{e^{2\pi i x}-1} &= \pi i\left(1 + \frac{2}{e^{2\pi i x}-1}\right) \\ &= \pi i + \frac{1}{x}\frac{2\pi i x}{e^{2\pi i x}-1} \\ &= \frac{1}{x} + \sum_{n>0} \frac{B_{2n}(2\pi i)^{2n}}{(2n!)} x^{2n-1}, \end{aligned}$$

and so the result follows. ∎

The values at odd integers are significantly less well understood. They are conjectured to be transcendental, and there is the standard conjecture about their algebraic independences.

Conjecture 1.2.2. The collection $\{\pi^2, \zeta(3), \zeta(5), \zeta(7), \ldots\}$ are algebraically independent, i.e. they are no polynomial relations with rational coefficients among any subset thereof.

However little is known, even about the irrationality of $\zeta(2n+1)$.

Theorem 1.2.3 (Apery [2])**.** *$\zeta(3)$ is irrational.*

Theorem 1.2.4 (Zudilin [12])**.** *At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.*

Theorem 1.2.5 (Ball-Rivoal, [3])**.** *The dimension of the $\mathbb{Q}$-vector space $\langle 1, \zeta(3), \ldots, \zeta(2n+1)\rangle_{\mathbb{Q}}$ is bounded below by $\frac{1}{3}\log(2n+1)$.*

While there have been some improvements to the lower bound of Ball-Rivoal, and some more elementary proofs of Zudilin-type results, this is largely the current state of knowledge. In general, it is unclear how to progress. However, if we consider the problem motivically, replacing the zeta values by formal analogues, satisfying only relations of "geometric" origin, all the standard conjectures become trivial. Unfortunately, the motivic proof is insufficient to prove much at the level of numbers, it still provides evidence that we should be optimistic.

1.2.2 The algebra of multiple zeta values

While values of the Riemann zeta function are conjectured to be algebraically independent, if we instead consider multiple zeta values, we obtain a rich algebraic structure.

Definition 1.2.6. For a sequence of positive integers $(k_1, \ldots, k_r)$ with $k_r > 1$, we define the associated multiple zeta value (MZV) to be the multisum

$$\zeta(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}}.$$

We define the *weight* of a multiple zeta value to be $k_1 + \cdots + k_r$ and the *depth* to be r.

Remark 1.2.7. Note that some authors us the reverse convention, requiring $k_1 > 1$ and summing over $n_1 > \cdots n_r > 0$. This produces the same set of numbers, but indexed with reversed tuples.

Lemma 1.2.8. *For $k_r > 1$, $\zeta(k_1, \ldots, k_r)$ converges.*

Proof. Note that it suffices to prove this for $\zeta(1, 1, \ldots, 1, 2)$. Using the fact that

$$\sum_{0<m<n} \frac{1}{m} \leq 1 + \log(m)$$

As such, we have the following upper bound

$$\zeta(1, 1, \ldots, 2) = \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1 \ldots n_{r-1} n_r^2} \leq \sum_{0<n} \frac{1}{n^2} \left(1 + \log(n)\right)^{r-1}$$

which clearly converges. ☐

Proposition 1.2.9. *The $\mathbb{Q}$-vector space generated by MZVs is a $\mathbb{Q}$-algebra.*

We delay the proof of this until the following subsection. However, as a $\mathbb{Q}$-algebra, we find many relations among MZVs.

Example 1.2.10.

$$\begin{aligned}\zeta(1,2) &= \zeta(3)\\ \zeta(2)\zeta(3) &= \zeta(2,3)+\zeta(3,2)+\zeta(5)\\ \zeta(2)^2 &= 4\zeta(1,3)+2\zeta(2,2)\\ &= \zeta(2,2)+\zeta(4)\end{aligned}$$

Note that, while these relations are not homogeneous for depth, they are for weight.

Conjecture 1.2.11. The algebra of MZVs is weight graded.

This is a challenging conjecture already: verifying it would imply the linear independence of all single zeta values. However, it again is motivically simple. Another open problem is to describe all relations among multiple zeta values (or even all motivic relations). There are a number of conjecturally complete candidates here:

- The (extended) double shuffle relations - a natural combinatorial double algebra structure
- The associator relations - a set of relations with connections to conformal field theory and knot theory
- The confluence relations - a set of relations arising from limits of polylogarithms

MZVs and their relations are not just interesting from a purely number theoretic point of view. They arise naturally as amplitudes in CFT and QFT [5], their relations can be used to construct universal knot invariants [4], and they are related to a group scheme through which the absolute Galois group factors [8].

We have a number of standard conjectures and results, some of which we will prove in the following lectures.

Conjecture 1.2.12. MZVs are graded by weight, and, letting d_n be the dimension of the weight n graded piece, we have

$$\sum_n \geq 0 d_n x^n = \frac{1}{1-x^2-x^3}.$$

Proposition 1.2.13. *Let D_n be the coefficient of x^n in $\frac{1}{1-x^2-x^3}$. Then D_n is an upper bound for the dimension of the weight n graded piece of the algebra of MZVs.*

Conjecture 1.2.14. MZVs of the form $\zeta(k_1,\ldots,k_r)$, $k_i \in \{2,3\}$ form a $\mathbb{Q}$-basis for the algebra of MZVs.

Proposition 1.2.15. *MZVs of the form $\zeta(k_1,\ldots,k_r)$, $k_i \in \{2,3\}$ form a $\mathbb{Q}$-spanning set for the algebra of MZVs.*

1.2.3 The double shuffle relations

The double shuffle relations are a simple set of relations coming from a double algebra structure on the $\mathbb{Q}$-vector space of MZVs. it consists of two sets of relations - the shuffle relations and the stuffle relations. We first consider the stuffle relations

Example 1.2.16. Consider the product $\zeta(2)\zeta(3)$. But dividing up the domain of summation, we find

$$\begin{aligned}\zeta(2)\zeta(3) &= \sum_{0<m}\sum_{0<n}\frac{1}{m^2n^3}\\ &= \sum_{0<m<n} + \sum_{0<n<m} + \sum_{0<m=n}\frac{1}{m^2n^3}\\ &= \zeta(2,3)+\zeta(3,2)+\zeta(5).\end{aligned}$$

By similarly dividing up the domain of summation, we obtain a decomposition of any product of MZVs as a sum. The precise definition may be given recursively as follows.

Definition 1.2.17. Let $\mathbb{Q}\langle y_1, y_2, \ldots\rangle$ be the $\mathbb{Q}$-vector space of noncommutative monomials in variables $\{y_n\}_{n>0}$. We define the stuffle product by

$$\begin{aligned}1\star w = w\star 1 &= w\\ y_iu\star y_jv &= y_i(u\star y_jv)+y_j(y_iu\star v)+y_{i+j}(u\star v)\end{aligned}$$

for any w, u, v a monomial in $\{y_n\}$, and extended by linearity.

Theorem 1.2.18. *Defining $\zeta : \mathbb{Q}\langle y_1, y_2, \ldots\rangle \to \mathbb{C}$ to be the $\mathbb{Q}$-linear extension of*

$$\zeta(y_{k_1}\ldots y_{k_r}) := \zeta(k_1,\ldots,k_r)$$

we have that

$$\zeta(u)\zeta(v) = \zeta(u\star v).$$

To discuss the shuffle relations, we must first write MZVs as iterated integrals. We will give a more precise definition later in the course.

Theorem 1.2.19. *For an MZV $\zeta(k_1,\ldots,k_r)$ of weight N, we may write this as the iterated integral*

$$\int_{0<t_1<\cdots<t_N<1}\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{k_1}}{t_{k_1}}\frac{dt_{k_1+1}}{1-t_{k_1+1}}\frac{dt_{k_1+2}}{t_{k_1+2}}\cdots\frac{dt_{k_1+\cdots k_{r-1}+1}}{1-t_{k_1+\cdots k_{r-1}+1}}\frac{dt_{k_1+\cdots k_{r-1}+2}}{t_{k_1+\cdots k_{r-1}+2}}\cdots\frac{dt_N}{t_N}$$

where the integrand consists of a $\frac{dt}{1-t}$ followed by k_1-1 copies of $\frac{dt}{t}$, a $\frac{dt}{1-t}$ followed by k_2-1 copies of $\frac{dt}{t}$, ….

Example 1.2.20. Consider the iterated integral

$$\begin{aligned}\int_{0<x<y<z<1} \frac{dx}{1-x}\frac{dy}{1-y}\frac{dz}{z} &= \sum_{m>0}\int_{0<y<z<1} \frac{y^m}{m}\frac{dy}{1-y}\frac{dz}{z} \\ &= \sum_{m>0}\sum_{n>0}\frac{1}{m}\frac{z^{m+n}}{m+n}\frac{dz}{z} \\ &= \sum_{m,n>0}\frac{1}{m}\frac{1}{(m+n)^2} \\ &= \sum 0<m<n\frac{1}{mn^2} = \zeta(1,2).\end{aligned}$$

The shuffle relations are given by subdivision of the domain of integration of a product.

Example 1.2.21. Consider the product $\zeta(2)^2$. We can write

$$\begin{aligned}\zeta(2)^2 =& \int_{0<x_1<x_2<1}\frac{dx_1dx_2}{(1-x_1)x_2}\int_{0<y_1<y_2<1}\frac{dy_1dy_2}{(1-y_1)y_2} \\ =& \int_{0<x_1<x_2<y_1<y_2<1} + \int_{0<x_1<y_1<x_2<y_2<1} + \int_{0<x_1<y_1<y_2<x_2<1} \\ &+ \int_{0<y_1<x_1<x_2<y_2<1} + \int_{0<y_1<x_1<y_2<x_2<1} + \int_{0<y_1<y_2<x_1<x_2<1} \\ =&\zeta(2,2)+\zeta(1,3)+\zeta(1,3)+\zeta(1,3)+\zeta(1,3)+\zeta(2,2) \\ =&2\zeta(2,2)+4\zeta(1,3).\end{aligned}$$

Definition 1.2.22. Let $\mathbb{Q}\langle e_0 e_1\rangle$ be the $\mathbb{Q}$-vector space of noncommutative monomials in variables $\{e_0, e_1\}$. We define the stuffle product by

$$\begin{aligned}1 ⧢ w = w ⧢ 1 &= w \\ xu\star yv &= x(u ⧢ yv) + y(xu ⧢ v)\end{aligned}$$

for any w, u, v a monomial in $\{e_0, e_1\}$, and extended by linearity.

Theorem 1.2.23. *Defining* $\zeta : e_1\mathbb{Q}\langle e_0, e_1\rangle e_0 \to \mathbb{C}$ *to be the* $\mathbb{Q}$*-linear extension of the map sending* $w_1w_2\dots w_n$ *to the iterated integral whose* i^{th} *differential form is* $\frac{dt}{1-t}$ *if* $w_i = e_1$ *and* $\frac{dt}{t}$ *otherwise. Then*

$$\zeta(u)\zeta(v) = \zeta(u ⧢ v).$$

1.3 Exercises

1. Justify (non-rigorously) the equivalence of definitions 1.1.1 and 1.1.4.
2. Using beta integrals, prove $\Gamma(p/q)^q \in \mathcal{P}$. Relate π as the area of the unit circle to $\pi = \Gamma(1/2)^2$.
3. Prove that MZVs are given by the iterated integral as claimed.
4. Convince yourself that the recursive definition of the stuffle and shuffle relations correspond to the illustrated subdivision of domains of summation/integration.
5. Prove in two ways $\zeta(1,2) = \zeta(3)$.

Lecture 2

2.1 Periods from cohomology

In this section, we give yet another definition of a period, this time in terms of a compariso between cohomology theories of varieties. This will allow us to define motivic periods in terms of Deligne motives [6]

2.1.1 Betti cohomology

Let M be a topological space. We define a singular n-chain to be a continuous map

$$\sigma : \Delta^n_{st} \to M$$

where

$$\Delta^n_{st} := \{(t_0, \ldots, t_n) \in \mathbb{R}^n \mid \sum_{i=0}^{n} t_i,\ t_j \geq 0\}$$

is the standard n-dimensional simplex. Let $C_n(M)$ be the free $\mathbb{Q}$-vector space generated by (singular) n-chains. The collection $\{C_n(M)\}_{n\geq 0}$ forms a chain complex when equipped with the differentials

$$\partial_n : C_n(M) \to C_{n-1}(M)$$

defined as follows. Let $\delta^n_i : \Delta^{n-1}_{st} \to \Delta^n_{st}$ be the i^{th} face map

$$(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

We then define

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta^n_i).$$

We then define the ith Betti homology group by

$$H_i(M, \mathbb{Q}) := \ker\partial_n / \mathrm{im}\partial_{n+1}$$

Example 2.1.1. Let $M = \mathbb{C}^\times$. Then 0-chains consist of linear combinations of points, 1-chains consist of linear combinations of paths, and 2-chains consist of linear combinations of simple connected, bounded 2 dimensional open sets (viewed as a subspace of $\mathbb{R}^2$). One can easily check

$$H_0(M, \mathbb{Q}) \cong \mathbb{Q}$$
$$H_1(M, \mathbb{Q}) = \mathbb{Q}[\gamma_0]$$
$$H_n(M, \mathbb{Q}) = 0 \text{ for } n \geq 2$$

where γ_0 is a closed loop around 0.

Remark 2.1.2. One often speaks of Betti cohomology, which is defined to be the dual space to Betti homology:

$$H^i(M,\mathbb{Q}) := \mathrm{Hom}_{\mathbb{Q}}(H_i(M,\mathbb{Q}),\mathbb{Q}).$$

We can also define a more general notion of relative homology. Let $N \subset M$ be a subspace of M and let $\iota : N \to M$ be the inclusion map. We define the complex of relation chains by

$$C_n(M,N) := C_n(M) \oplus C_{n-1}(N)$$

equipped with the differential

$$\partial_n(a,b) := (\partial_n a - \iota_\star b, -\partial_{n-1}b).$$

We define the relative homology

$$H_n(M,N) := \ker\partial_n/\mathrm{im}\partial_{n+1}.$$

Remark 2.1.3. One can show that this complex is quasi-isomorphic to the complex $C_\bullet(M)/C_\bullet(N)$ equipped with the differential induced by that of $C_\bullet(M)$. As such, we may think of elements of $H_n(M,N)$ as represented by n-chains with boundary in N.

Example 2.1.4. Taking $M = \mathbb{C}^\times$, $N = \{1,2\}$, we find

$$H_1(M,N) = \mathbb{Q}[\gamma_0] \oplus \mathbb{Q}[\gamma_{1,2}]$$
$$H_n(M,N) = 0 \text{ for all } n \geq 1$$

where γ_0 is a loop around 0, and $\gamma_{1,2}$ is the straight line path from 1 to 2.

2.1.2 de Rham cohomology

In order to keep things simple, we will only consider the case where X is a smooth, affine variety over $\mathbb{Q}$. The general case follows upon replacing Ω^1_X with the corresponding sheaf of differentials and taking hypercohomology.

Assume $X = \mathrm{Spec}(A)$ for a $\mathbb{Q}$-algebra $A = \mathbb{Q}[x_1,\ldots,x_n]/(f_1,\ldots,f_m)$. We define the A-module of differentials

$$\Omega^1_X := (\bigoplus_{i=1}^n A dx_i)/(\bigoplus_{j=1}^m A df_j)$$

where $df := \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$ for any $f \in A$. We further define $\Omega^p_X := \bigwedge^p \Omega^1_X$, and $\Omega^0_X := A$.

This forms a chain complex with differential given by

$$d_k(f dx_{i_1} \wedge \cdots \wedge dx_{i_k} := df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and we define the de Rham cohomology as the $\mathbb{Q}$-vector space given by

$$H^k_{dR}(X) := \ker d_k/\mathrm{im} d_{k-1}.$$

When it is clear from context, we will suppress the subscripts and merely write d for all differentials.

Example 2.1.5. Let $X = \mathbb{G}_m$ be the algebraic variety with R points given by the units of R, i.e. $X = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}])$. The de Rham complex is then given by

$$\mathbb{Q}[t, t^{-1}] \to \mathbb{Q}[t, t^{-1}]dt$$

Having $d(t^n) = nt^{n-1}dt$, we find

$$H^0_{dR}(X) \cong \mathbb{Q}$$
$$H^1_{dR}(X) = \mathbb{Q}\left[\frac{dt}{t}\right]$$
$$H^n_{dR}(X) = 0 \text{ for all } n \geq 2.$$

Just like the case of Betti cohomology, we can define a relative version of de Rham cohomology, elements of which may be viewed as closed differential forms whose restriction to a subvariety is exact. To be precise, for $\iota : Y \to X$ a subvariety of X we define

$$\Omega^k(X, Y) := \Omega^k(X) \oplus \Omega^{k-1}(Y)$$

and define the differential

$$d_k(\alpha, \beta) := (d\alpha, \iota^\star\alpha - d\beta).$$

The relative cohomology groups are then defined by

$$H^k_{dR}(X, Y) := \ker d_k / \mathrm{im} d_{k-1}.$$

Example 2.1.6. Let $X = \mathbb{G}_m$, and $Y = \{1, 2\}$. The de Rham complex is then given by

$$\mathbb{Q}[t, t^{-1}] \to \mathbb{Q}[t, t^{-1}]dt \oplus \mathbb{Q} \oplus \mathbb{Q}$$

where $d(f) = (df, f(1), f(2))$. This map is injective, and so $H^0_{dR}(X, Y) = 0$, and $H^n_{dR}(X, Y) = 0$ for all $n \geq 2$. To compute $\mathrm{im} d_0$, note that it is spanned by

$$\{(nt^{n-1}dt, 1, 2^n\}_{n \neq 0} \cup \{(0, 1, 1)\}.$$

The kernel of d_1 is $\mathbb{Q}[t, t^{-1}] \oplus \mathbb{Q} \oplus \mathbb{Q}$, and so one can readily verify that

$$H^1_{dR}(X, Y) = \mathbb{Q}[(dt/t, 0, 0)] \oplus \mathbb{Q}[(0, 1, 0)].$$

One can furthermore check that $[(0, 1, 0)] = [(dt, 0, 0)]$.

2.1.3 The comparison isomorphism

Theorem 2.1.7 (Grothendieck [1])**.** *For X a smooth variety over $\mathbb{Q}$, there exists a canonical isomorphism*

$$\mathrm{comp}_{dR,B} : H^k_{dR}(X) \otimes_{\mathbb{Q}} \mathbb{C} \to H^k_B(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

For affine X, this is the isomorphism induced by the perfect pairing

$$H^k_{dR}(X) \otimes_{\mathbb{Q}} H^k_B(X(\mathbb{C}), \mathbb{Q}) \to \mathbb{C},$$
$$\omega \otimes \sigma^\vee \mapsto \int_\sigma \omega.$$

There is also a relative version of this isomorphism

$$\mathrm{comp}_{dR,B} : H^k_{dR}(X,Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^k_B(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

given on affine X by integration:

$$(\omega_X, \omega_Y) \otimes (\sigma_X^\vee, \sigma_Y^\vee) \mapsto \int_{\sigma_X} \omega_X + \int_{\sigma_Y} \omega_Y$$

Definition 2.1.8. The set of periods is the image of this pairing as (X,Y) vary across smooth varieties. A period matrix is given by the matrix of $\mathrm{comp}_{dR,B}$ in a chosen basis.

Example 2.1.9. Let $X = \mathcal{G}_m$. Recall that $H^1_{dR} = \mathbb{Q}\left[\frac{dt}{t}\right]$ and $H^1_R\mathbb{Q}[\gamma_0^\vee]$. The period matrix is given by

$$\left(\int_{\gamma_0} \frac{dt}{t}\right) = (2\pi i)\,.$$

Example 2.1.10. Let $X = \mathcal{G}_m$, $Y = \{1,2\}$. The period matrix is given by

$$\begin{pmatrix} 1 & \log 2 \\ 0 & 2\pi i \end{pmatrix}$$

We can think of these periods as being encoded by tuples (X, Y, ω, γ), where $\omega \in H^\bullet_{dR}(X,Y)$, $\gamma \in H_\bullet(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ via the following results

Lemma 2.1.11. *A period (X, Y, ω, γ) depends only on the equivalences classes of ω and γ in (co)homology.*

Proof. We leave the case of relative homology as an exercise to the reader. Suppose $[\omega_1] = [\omega_2] \in H^k_{dR}(X)$. Then there exists $\omega_+ \in H^{k-1}_{dR}(X)$ such that $d\omega_+ = \omega_1 - \omega_2$. Hence

$$\int_\gamma \omega_1 - \int_\gamma \omega_2 = \int_\gamma d\omega_+ = \int_{\partial\gamma} \omega_+ = 0$$

as $\partial\gamma = 0$. Similarly, if $[\gamma_1] = [\gamma_2]$, there exists γ_+ such that $\gamma_1 - \gamma_2 = \partial\gamma_+$. Hence

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial\gamma_+} \omega = \int_{\gamma_+} d\omega = 0$$

as $d\omega = 0$. □

Theorem 2.1.12. *The set of periods of Definition 2.1.8 is equal to the set of naive periods of Definition 1.1.1.*

Proof. The reader may find the proof of this result under Theorem 12.2.1 in [9]. □

Remark 2.1.13. Huber's proof in fact shows that all periods may be obtains as integrals of forms of top degree, and that Y may be assumed to be a normal crossings divisor.

2.2 Exercises

1. Verify the cohomology computations given in the examples.
2. Compute the first Betti cohomology group of the elliptic curve $E : y^2 = x(x-1)(x-\lambda)$.
3. Compute the first de Rham cohomology group of E and hence the period matrix associated to your chosen basis
4. Proof Lemma 2.1.11 for relative cohomology groups.

Lecture 3

3.1 Neutral Tannakian categories

In this section, we will run through a series of definitions leading up to that of a Tannakian category. However, due to time constraints in the lecture, definitions and results on affine group schemes will be delayed to the next chapter.

3.1.1 Rigid tensor categories

Definition 3.1.1. A $\otimes$-category is a category $\boldsymbol{C}$ with a bifunctor $\otimes : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$. We call a $\otimes$-category ACU if the following properties hold:

1. There exists a functorial isomorphism $\Psi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$.
2. There exists a functorial isomorphism $\Phi_{X,Y} : X \otimes Y \to Y \otimes X$.
3. There exists an object $\mathbb{1} \in \mathrm{Ob}\boldsymbol{C}$ and functorial isomorphisms

$$\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$$

satisfying the MacLane coherence conditions e.g. the isomorphism

$$X \otimes (Y \otimes (Z \otimes W)) \cong ((X \otimes Y) \otimes Z) \otimes W$$

should be independent of the order of rebracketing:

$$\Psi_{X\otimes Y,Z,W} \circ \Psi_{X,Y,Z\otimes W} = \Psi_{X,Y,Z} \otimes \mathrm{id}_W \circ \Psi_{X,Y\otimes Z,W} \circ \mathrm{id}_X \otimes \Psi_{Y,Z,W}$$

Definition 3.1.2. A functor $F : \boldsymbol{C} \to \boldsymbol{D}$ of $\otimes$-categories is called a $\otimes$-functor if there exist functorial isomorphisms

$$c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y).$$

F is called ACU if it is compatible with the associativity and commutativity isomorphisms in the obvious way, and there exists an isomorphism

$$a_F : \mathbb{1}_{\boldsymbol{D}} \cong F(\mathbb{1}_{\boldsymbol{C}})$$

satisfying the obvious compatibilities.

Definition 3.1.3. A $\otimes$-morphism between $\otimes$-functors $\mu : F \to G$ is a natural transformation such that

$$\mu_{X\otimes Y} \circ c_{X,Y} = d_{X,Y} \circ (\mu_X \otimes \mu_Y)$$

for all X, Y objects of $\boldsymbol{C}$. If F, G are ACU, we call μ unitial if $a_G = \mu_{\mathbb{1}} \circ a_F$. We denote the set of (unital) $\otimes$-morphism from F to G by $\mathrm{Hom}^{\otimes,(1)}(F, G)$.

Definition 3.1.4. We has a $\otimes$-category $\boldsymbol{C}$ has Hom-objects if for all $X, Y \in \mathrm{Ob}\boldsymbol{C}$ there exists an object $\boldsymbol{Hom}(X, Y)$ and an isomorphism, functorial in Z

$$\mathrm{Hom}(Z \otimes X, Y) \cong \mathrm{Home}(Z, \boldsymbol{Hom}(X, Y)).$$

Denote by $\mathrm{ev}_{X,Y}$ the morphism corresponding to $\mathrm{id}_{\mathrm{Hom}(X,Y)}$ under this isomorphism.

Note $\boldsymbol{Hom}$ is unique only up to isomorphism. Furthermore note that, to every $g : Z \otimes X \to Y$, there exists a unique $f : Z \to \mathrm{Hom}(X, Y)$ such that $g = \mathrm{ev}_{X,Y} \circ (f \otimes \mathrm{id}_X)$.

Note also that we must have

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(\mathbb{1} \otimes X, Y) \cong \mathrm{Hom}(\mathbb{1}, \mathrm{Hom}(X, Y)).$$

We may think of elements of the last set as "points" of $\mathrm{Hom}(X, Y)$, and so Hom-objects define some sort of inclusion of Hom-sets into our category.

Indeed, in any category $\boldsymbol{C}$ which contains its Hom-sets, these may be taken as Hom-objects. For example, in $\boldsymbol{Mod}_R$, R a commutative ring, we have that $\mathrm{Hom}_R(X, Y)$ is an R-module and

$$\mathrm{Hom}_R(Z \otimes X, Y) \cong \mathrm{Hom}_R(Z, \mathrm{Hom}_R(X, Y).$$

We furthemore have that $\mathrm{ev}_{X,Y}(f \otimes x) = f(x)$, hence calling it an evaluation morphism.

Definition 3.1.5. We define the dual of X to be the object $X^\vee := \mathrm{Hom}(X, \mathbb{1})$.

The easiest example of a dual object to keep in mind is that of the dual vector space $V^\vee := \mathrm{Hom}_k(V, k)$ in $\boldsymbol{Vec}_k$.

Lemma 3.1.6. *For every object X in a $\otimes$-category $\boldsymbol{C}$ with Hom-objects,*

$$\mathrm{Hom}(\mathbb{1}, X) \cong X.$$

Proof. The Hom object $\mathrm{Hom}(\mathbb{1}, X)$ is defined up to isomorphism by the condition

$$\mathrm{Hom}(Z, \mathrm{Hom}(\mathbb{1}, X)) \cong \mathrm{Hom}(Z \otimes \mathbb{1}, X)$$

functorially for all Z. However, $\mathrm{Hom}(Z \otimes \mathbb{1}) \cong \mathrm{Hom}(Z, X)$, functorially in X. Hence X is another representing object of $\mathrm{Hom}(\mathbb{1}, X)$, and is thus isomorphic. $\square$

Lemma 3.1.7. *For all X, Y objects of a $\otimes$-category $\boldsymbol{C}$ with Hom-objects, there exists a morphism*

$$X^\vee \otimes Y \to \mathrm{Hom}(X, Y)$$

Proof. We have that

$$\mathrm{Hom}(X^\vee \otimes Y, \mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X^\vee \otimes Y \otimes X, Y) \cong \mathrm{Hom}(\mathrm{Hom}(X, \mathbb{1}) \otimes X \otimes \mathrm{Hom}(\mathbb{1}, Y) \otimes \mathbb{1}, \mathbb{1} \otimes Y)$$

The latter Hom-set contains $\mathrm{ev}_X \otimes \mathrm{ev}_{\mathbb{1},Y}$, and hence the former is non-empty. $\square$

We next define a rigid category. First, we require a lemma.

Lemma 3.1.8. *Let $\boldsymbol{C}$ be a $\otimes$-category with Hom-objects. Then there exist canonical morphisms*

$$i_X : X \to X^{\vee\vee}$$
$$\tau_{X,Y,X',Y'} : \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(X',Y') \to \mathrm{Hom}(X \otimes X', Y \otimes Y')$$

Proof. Let i_X be the morphism corresponding to ev_X under the isomorphim

$$\mathrm{Hom}(X, X^{\vee\vee}) \cong \mathrm{Hom}(X \otimes X^{\vee}, \mathbb{1}) \cong \mathrm{Hom}(X^{\vee} \otimes X, \mathbb{1})$$

and let $\tau_{X,Y,X',Y'}$ be the morphism corresponding to $\mathrm{ev}_{X,Y} \otimes \mathrm{ev}_{X',Y'}$ under the isomorphism

$$\begin{aligned} &\mathrm{Hom}(\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(X',Y'), \mathrm{Hom}(X \otimes X', Y \otimes Y') \\ \cong &\mathrm{Hom}(\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(X',Y') \otimes X \otimes X', Y \otimes Y') \\ \cong &\mathrm{Hom}(\mathrm{Hom}(X,Y) \otimes X \otimes \mathrm{Hom}(X',Y') \otimes X', Y \otimes Y') \end{aligned}$$

□

Definition 3.1.9. A rigid category is an ACU $\otimes$-category $\boldsymbol{C}$ with Hom-objects such that the morphisms i_X, $\tau_{X,Y,X',Y'}$ of Lemma 3.1.8 are isomorphisms. A rigid functor is a unital $\otimes$-functor between rigid categories.

Lemma 3.1.10. *For all X, Y objects of a rigid category $\boldsymbol{C}$, there exists an isomorphism*

$$X^{\vee} \otimes Y \to \mathrm{Hom}(X,Y)$$

Proof. It suffices to show that there is an isomorphism, functorial in Z,

$$\mathrm{Hom}(Z \otimes X, Y) \cong \mathrm{Hom}(Z, X^{\vee} \otimes Y)$$

We have that

$$\begin{aligned} Hom(Z, X^{\vee} \otimes Y) &\cong \mathrm{Hom}(Z, \mathrm{Hom}(X\mathbb{1}) \otimes \mathrm{Hom}(\mathbb{1}, Y)) \\ &\cong \mathrm{Hom}(Z, \mathrm{Hom}(X \otimes \mathbb{1}, \mathbb{1} \otimes Y)) \end{aligned}$$

using that $\tau_{X,\mathbb{1},\mathbb{1},Y}$ is an isomorphism. But this is functorially isomorphic to

$$\mathrm{Hom}(Z, \mathrm{Hom}(X,Y) \cong \mathrm{Hom}(Z \otimes X, Y).$$

Hence, the result follows. □

Using this isomorphism, one may reformulate the definition of a rigid category in terms of properties of dual objects.

Proposition 3.1.11. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be rigid categories and $F, G : \boldsymbol{C} \to \boldsymbol{D}$ be rigid functors. Then*

$$F(\mathrm{Hom}_{\boldsymbol{C}}(X,Y)) \cong \mathrm{Hom}_{\boldsymbol{D}}(F(X), F(Y))$$

and

$$\mathrm{Hom}^{\otimes,1}(F,G) = \mathrm{Isom}^{\otimes}(F,G).$$

Proof. To show the first isomorphism, it is sufficient to show that $F(X^\vee) \cong F(X)^\vee$, by Lemma 3.1.10. One can show that $X^\vee$ is determined uniquely up to isomorphism as a pair $(Y, \mathrm{ev} : Y \otimes X \to \mathbb{1})$ for which we can find an $\epsilon : \mathbb{1} \to X \otimes Y$ such that

$$X \cong \mathbb{1} \otimes X \xrightarrow{\epsilon \otimes \mathrm{id}} (X \otimes Y) \otimes X \cong X \otimes (Y \otimes X) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} X$$

and

$$Y \cong Y \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \epsilon} Y \otimes (X \otimes Y) \cong (Y \otimes X) \otimes Y \xrightarrow{\mathrm{ev} \otimes id} Y$$

are the identity maps. This property is clearly preserved by a rigid functor, and hence F preserves duals.

To see that every unitial $\otimes$-morphism of rigid functors μ is an isomorphism, note first that any morphism $f : X \to Y$ in a rigid category induces a transpose isomorphism ${}^t f : Y^\vee \to X^\vee$. We define $\lambda : G \to F$ to be the unique morphism such that

$$\begin{array}{ccc} F(X^\vee) & \xrightarrow{\mu_{X^\vee}} & G(X^\vee) \\ \downarrow\wr & & \downarrow\wr \\ F(X)^\vee & \xrightarrow{{}^t\lambda_X} & G(X)^\vee \end{array}$$

λ can easily be checked to be inverse to μ. □

Examples of rigid categories include $\boldsymbol{Vec}_k, \boldsymbol{Mod}_R, A - \boldsymbol{Alg}, \boldsymbol{Rep}_k(G)$ for k a field, R a commutative ring, A a k-algebra, G a group or affine group scheme.

Remark 3.1.12. In a rigid category, we have an morphism

$$\mathrm{Hom}(X, X) \cong X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}$$

Applyin the functor $\mathrm{Hom}(\mathbb{1}, -)$, we obtain a trace morphism

$$\mathrm{Tr}_X : \mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$$

satisfying

$$\mathrm{Tr}_{X \otimes Y}(f \otimes f) = \mathrm{Tr}_X(f)\,\mathrm{Tr}_Y(g).$$

We can use the trace functor to define a rank, $\mathrm{rank}(X) := \mathrm{Tr}_X(\mathrm{id}_X)$. This gives the usual rank in categories of modules, representations, vector spaces, etc.

Definition 3.1.13. An abelian $\otimes$-category is a category that is both an ACU $\otimes$-category and an abelian abelian, such that $\otimes$ is a biadditive functor.

Remark 3.1.14. If $\boldsymbol{C}$ is an abelian tensor category, then $R := \mathrm{End}(\mathbb{1})$ is a ring that acts via $X \cong \mathbb{1} \otimes X$, on each object of $\boldsymbol{C}$. The action of R commutes with endomorphism of X and the category $\boldsymbol{C}$ is R-linear, as is the tensor product $\otimes$.

Proposition 3.1.15. *An abelian category that is rigid is an abelian tensor category. Specifically, $\otimes$ is biadditive, commutes with direct and inverse limits in each variable, and is exact in each variable.*

Proof. The functors $- \otimes Y$ and $\mathrm{Hom}(Y, -)$ are, by definition, adjoint. Thus $- \otimes Y$ commutes with direct limits and is additive. They are also adjoint in the oppositive category, and so it commutes with inverse limits. The result then follows. □

We call such a category a rigid abelian category. We shall not prove the following, but merely mention it as a method of determining when a category is a rigid abelian category.

Proposition 3.1.16. *Let k be a field, and let $\boldsymbol{C}$ be a k-linear abelian category equippied with a k-bilinear functor $\otimes : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$. Suppose there exists a faithful, exact, k-linear functor $F : \boldsymbol{C} \to \boldsymbol{Vec}_k$ and functorial isomorphisms Ψ, Φ as in Definition 3.1.1, with the following properties:*

1. $F \circ \otimes = \otimes \circ (F \times F)$,
2. $F \circ \Psi_{X,Y,Z}$ *is the standard associativity isomorphism in* $\boldsymbol{Vec}_k$,
3. $F \circ \Phi_{X,Y}$ *is the standard commutativity isomorphism in* $\boldsymbol{Vec}_k$,
4. *There exists an object* $\mathbb{1}$ *in* $\boldsymbol{C}$ *and functorial isomorphisms*
$$\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$$
such that $k \cong \mathrm{End}(\mathbb{1})$ *and* $dimF(\mathbb{1})$.
5. *If* $dimF(X) = 1$ *, there exists an object* X^{-1} *such that* $X \otimes X^{-1} \cong \mathbb{1}$.

Then $\boldsymbol{C}$ *is a rigid abelian category.*

This proposition can be used to show may categories to be rigid abelian. In particular, this shows $\boldsymbol{Rep}_k(G)$ is rigid abelian for any affine group scheme G over k. An example of a rigid abelian category not of this form may be given by the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces where $\Phi_{V,W}(v \otimes w) := (-1)^{|v||w|} w \otimes v$. One can easily check that this is a rigid abelian category, but cannot be a category of representations, as the rank map is not non-negative.

3.1.2 Interlude: Tannaka duality

In order to break up a long series of definitions, and to help motivate why we have made such definitions, we will now discuss the end goal of this chapter and the following chapter: Tannaka duality.

Theorem 3.1.17. *Let $\boldsymbol{C}$ be a rigid abelian category over k such that $k = \mathrm{End}(\mathbb{1})$ and equipped with an exact, faithful, k-linear functor - called a fibre functor - $\omega : \boldsymbol{C} \to \boldsymbol{Vec}_k$. Then*

- *Let R be a commutative k-algebra, and $\phi_R : \boldsymbol{Vec}_k \to \boldsymbol{Mod}_R$ be the functor $V \to V \otimes_k R$. We define $\underline{Aut}^{\otimes}(\omega)$ to be the functor*
$$k - \boldsymbol{Alg} \to \boldsymbol{Grp}$$
$$R \mapsto \mathrm{Aut}^{\otimes}(\phi_R \circ \omega, \phi_R \circ \omega).$$
Then $\underline{Aut}^{\otimes}(\omega)$ is represented by an affine group scheme G.
- $\boldsymbol{C} \to \boldsymbol{Rep}_k(G)$ *is an equivalence of categories.*

We call such a category a neutral Tannakian category. We can see many of the properties of the group reflected in those of the category.

Proposition 3.1.18. *1. G is finite iff there exists $X \in \boldsymbol{Rep}_k(G)$ such that every object in $\boldsymbol{Rep}_k(G)$ is isomorphic to a subquotient of some $X^{\otimes n}$.*

2. *G is algebraic iff there exists $X \in \boldsymbol{Rep}_k(G)$ such that every object is isomorphic to a subquotient of $P(X, X^\vee)$ where $P \in \mathbb{N}[x, y]$.*

3. *If $f : G \to H$ is a homomorphism of group schemes, and ω^f is the corresponding functor $\boldsymbol{Rep}_k(H) \to \boldsymbol{Rep}_k(G)$, then f is faithfully flat iff ω^f is fully faithful and every subobject of $\omega^f(X)$ is isomorphic to the image of a subobject of X.*

4. *f is a closed immersion iff every object of $\boldsymbol{Rep}_k(G)$ is isomorphic to a subquotient of an object $\omega^f(X)$.*

5. *If k has characteristic 0, G is connected iff, for every X on which G acts non-triviallіy, the tensor subcategory generated by X is not stable under $\otimes$.*

6. *If G is a connected affine group scheme over a field of characteristic 0, then $\boldsymbol{Rep}_k(G)$ is semisimle iff G is pro-reductive.*

Example 3.1.19. Let $\boldsymbol{C}$ be the category of $\mathbb{Z}$-graded vector spaces over k with graded maps. Objects are collections $V = (V^n)$ of k-vector spaces V^n such that $\bigoplus_{n\in\mathbb{Z}} V^n$ is finite dimensional, and morphisms are collections of k-linear maps $(L_n : V^n \to W^n)$. We claim the group scheme $\underline{Aut}^{\otimes}(\omega)$ is isomorphic to the multiplicative group $\mathbb{G}^m$. An element of $\underline{Aut}^{\otimes}(\omega)(R)$ is a collection of R-linear isomorphisms $\{\eta_X : \omega(X) \otimes R \to \omega(X) \otimes R\}_{X\in \mathrm{Ob}\boldsymbol{C}}$ such that

$$\eta_{X\otimes Y} = \eta_X \otimes \eta_Y$$

and for every $\alpha : X \to Y$ in $\boldsymbol{C}$

$$\eta_Y \circ \omega(\alpha) = \omega_\alpha \circ \eta_Y.$$

Denote by $[n]$ the graded vector space (V^n) with $V^n = k$, $V^m = 0$ for all $m \neq n$, and let $\alpha : [n] \to W$ be the map taking $1 \mapsto w_0 \in W^n$. We must then have

$$\eta_W(w_0) = \omega(\alpha)\eta_{[n]}(1)$$

As $\eta_{[n]}$ is an isomorphism $R \to R$, we must have $\eta_{[n]}(1) = \lambda_n \in R^\times$. Then, by R-linearity, we get

$$eta_W(w_0) = \lambda_n w_0.$$

Note that λ_n is independent of α, w_0, and even W. Hence, we must have, for any collection $V = (V^n)$, any n, and any $v \in V^n$, we have

$$\eta_V(v) = \lambda_n v$$

for a collection λ_n determined by $\{\eta_{[n]}\}$. As we have $[m] \otimes [n] \cong [m + n]$, we must also have

$$\lambda_{m+n} = \lambda_m \lambda_n.$$

Hence

$$\lambda_n = \lambda_1^n$$

for any $\mathbb{Z}$. Every element $\eta \in \underline{Aut}^{\otimes}(\omega)(R)$ can be identified with a unique element of $R^\times$, and this correspondence is clearly an isomorphism.

Lecture 4

4.1 Affine group schemes

As mentioned in the last lecture, every (neutral) Tannakian category is equivalent to the category of representations of an affine group scheme, though we didn't quite have time to define an affine group scheme in detail. It will be useful to us later to discuss some properites and examples now. We will also prove a partial result on Tannaka duality.

Definition 4.1.1. Let k be a field. A Hopf algebra over k is a (unital) k-algebra A with three additional k-algebra morphisms

1. A coproduct $\Delta : A \to A \otimes A$,
2. A counit $\epsilon : A \to k$,
3. An antipode $S : A \to A$

satisfying

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id})\Delta,$$
$$\mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta,$$
$$\mu \circ (S \otimes id) \circ Delta = \eta \circ \epsilon,$$

where $\mu : A \otimes A \to A$ is multiplication, and $\eta : k \to A$ sends x to $x1_A$.

Definition 4.1.2. An affine group scheme over k is a functor $G = \mathrm{Spec}(A)$, for A a Hopf algebra.

$$G : \mathrm{Alg}_k \to \mathrm{Grp}$$
$$R \mapsto G(R) = \mathrm{Hom}(A, R)$$

Remark 4.1.3. We may alternatively define an affine group scheme G over k as a functor

$$G : \mathrm{Alg}_k \to \mathrm{Set}$$

equipped with a natural transformations

$$\mu : G \otimes G \to G$$
$$\iota : G \to G$$
$$\mathbb{1} : \mathrm{Hom}(k, -) \to G$$

satisfying axioms describing multiplication, inversion, and identity, as in a group. We leave the equivalence of these definitions as an exercise to the reader.

Remark 4.1.4. If A is finitely generated, we call G an algebraic affine group scheme.

Proposition 4.1.5. *The category of commutative Hopf algebas over k is equivalent to the category of affine group schemes*

This proposition tells us that A is uniquely determined up to isomorphism by $\mathrm{Spec}(A)$. In particular, given an affine group scheme G, we may recover A as $\mathcal{O}(G)$, the ring of regular functions on G.

Example 4.1.6. Let $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Q}[t])$. We may equip $\mathbb{Q}[t]$ with the structure of a Hopf algebra by defining

$$\begin{aligned}\Delta(t) &:= t \otimes 1 + 1 \otimes t \\ \epsilon(f(t)) &:= f(0) \\ (Sf)(t) &:= f(-t)\end{aligned}$$

Let R be a $\mathbb{Q}$-algebra. Then

$$\mathbb{G}_a(R) = \mathrm{Hom}(\mathbb{Q}[t], R) = R$$

as sets, since a morphism is determined by where we map t. Let $r, s \in \mathbb{G}_a(R)$. Then, $r \cdot s$ is defined by

$$(r \cdot s)(f(t)) = (r \otimes s)(\Delta(f(t))).$$

Since an element of $\mathbb{G}_a(R)$ is determined by its action on t, we have

$$(r \cdot s)(t) = (r \otimes s)(t \otimes 1 + 1 \otimes t) = r(t) + s(t)$$

and so $r \cdot s = r + s$. The identity element of $\mathbb{G}(R)$ is determined by $\epsilon(t) = 0$ to be 0, and S induces inversion, so $r^{-1} = -r$. Hence $\mathbb{G}_a(R)$ is the addition group underlying R.

Example 4.1.7. Let $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}])$, and equip $\mathbb{Q}[t, t^{-1}]$ with the structure of a Hopf algebra via

$$\begin{aligned}\Delta(t) &:= t \otimes t \\ \epsilon((f(t)) &:= f(1) \\ (Sf)(t) &:= f(t^{-1})\end{aligned}$$

One may easily verify that $\mathbb{G}_m(R) = R^{\times}$, the multiplicative group underlying R.

We may similarly obtain the functor that gives the group of roots of unity of of order N in R by $\mu_N = \mathrm{Spec}(\mathbb{Q}[t]/(t^N - 1))$, the general linear group $GL_n = \mathrm{Spec}(\mathbb{Q}[a_{i,j}, t]/(\det(a_{i,j})t - 1))$, and so on. It is also worth noting that a finite group H may be viewed as affine group schemes via $G = \mathrm{Spec}(\mathbb{Q}[H])$, equipping the group ring with the sturcture of a Hopf algebra via

$$\begin{aligned}\Delta e_\eta &:= \sum_{\rho\sigma=\eta} e_\rho \otimes e_\sigma, \\ \epsilon e_\eta &:= \delta_{\eta,\mathrm{id}}, \\ S(e_\eta) &:= e_{\eta^{-1}}.\end{aligned}$$

With this defintion $G(R) = H$ for all R without nilpotent elements

4.1.1 Comodules over coalgebras

As in the case of finite groups, it is often easier to think of an affine group scheme in terms of its representations.

Definition 4.1.8. A representation of an affine group scheme G is a k vector space V and a natural transformation

$$\eta : G \to \mathrm{End}_V$$

where $\mathrm{End}_V(R) := \mathrm{End}_{R-Mod}(V \otimes R)$, and such that η_R is a homomorphism for every R.

However, we more often think of represenations of an affine group scheme in terms of comodules over the corresponding Hopf algebra.

Definition 4.1.9. A coalgebra over a field k is a k-vector space C equipped with maps (Δ, ϵ) satisying

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id})\Delta,$$
$$\mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta.$$

Definition 4.1.10. Let C be a k-coalgebra. A (right) comodule over C is a vector space V and a linear map $\rho : V \to V \otimes C$ such that

$$(\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id},$$
$$(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho.$$

Proposition 4.1.11. *Let $G = \mathrm{Spec}(A)$ be an affine group scheme and let V be a vector space. There exists a (canonical) bijection between represenations of G on V and comodule structures over A on V.*

Proof. We will give two proofs of this claim. The first is more abstract, but establishes a canonical bijection. The second is more explicit, but depends on a choice of basis.

Let $\eta : G \to \mathrm{End}_V$ be a natural transformation. Then η_A maps $\mathrm{id}_A \in G(A) = \mathrm{Hom}(A, A)$ is mapped to an element of $\mathrm{End}_V(A) = \mathrm{End}(V \otimes A)$. Let us denote by ρ the restriction of $\eta_A(\mathrm{id}_A)$ to a map $V \to V \otimes A$. Let g be an element of $G(R)$.

Note that ρ uniquely determines $\eta_A(\mathrm{id}_A)$ as the unique A-linear extension of ρ to $V \otimes A$. This in turn uniquely determines $\eta_R(g)$, as the unique R-linear map such that

$$\eta_R(g) \circ (\mathrm{id}_V \otimes g) = (\mathrm{id}_V \otimes g) \circ \eta_A(a).$$

Hence η is uniquely determined by ρ. The same argument applies to any k-linear map $\rho : V \to V \otimes A$, establishing a bijection between natural transformations $\eta : G \to \mathrm{End}_V$ and k-linear maps $\rho : V \to V \otimes A$.

It remains to show that η defines a representation if and only if ρ defines a comodule. First consider the condition that $\eta_k(1_{G(k)}) = \mathrm{id}_{V \otimes k}$. As $1_{G(k)} = \epsilon$, this is equivalent to

$$(\mathrm{id}_V \otimes epsilon) \circ \rho = \mathrm{id}$$

which is precisely the co-unit condition. Next consider the condition $\eta_R(g)\eta_R(h) = \eta_R(gh)$. By definition, $\eta_R(gh)$ acts on V by

$$(\mathrm{id} \otimes (g \otimes h)) \circ (\mathrm{id} \otimes \Delta) \circ \rho$$

while $\eta_R(g)\eta_R(h)$ acts on V by

$$(\mathrm{id} \otimes (g \otimes h)) \circ (\rho \otimes \mathrm{id}) \circ \rho.$$

These are equal for all g, h if and only if the comodule coassociativity condition holds. Hence η is a represenation if and only if ρ is a comodule.

For the second proof, let $(e_i)_{i \in I}$ be a basis for V, allowing us to identify End_V with the matrix algebra $GL_{|I|}$, and natural transformations

$$\eta : G \to \mathrm{End}_V$$

with matrices with entries in A (viewed as functions on G via evaluation):

$$\eta_R(g) = (\eta_{i,j,R}(g)) textforg \in G(R).$$

We require that η_R be a homomorphism, which occurs if and only if

$$\eta_{i,j,R}(gh) = \sum_{k \in I} \eta_{i,k,R}(g)\eta_{k,j,R}(h)$$

for all $g, h \in G(R)$, and $\eta_{i,j,R}(1) = \delta_{i,j}$.

Now, consider a k-linear map $\rho : V \to V \otimes A$. This is equivalent to giving a matrix $(\eta_{i,j})$ of elements of A:

$$\rho(e_j) = \sum_{i \in I} e_i \otimes \eta_{i,j}.$$

We have that ρ defines a comodule if and only if

$$\Delta(\eta_{i,j}) = \sum_{k \in I} \eta_{i,k} \otimes \eta_{k,j}$$

and $\epsilon(\eta_{i,j}) = \delta_{i,j}$.

Then since

$$Delta(\eta_{i,j,R})(g \otimes h) = (\eta_{i,j,R})(gh)$$

, and

$$\sum_{k \in I} (\eta_{i,k,R} \otimes \eta_{k,j,R})(g \otimes h) = \sum_{k \in I} \eta_{i,k,R}(g)\eta_{k,j,R}(h),$$

we see that η is a homomorphism if and only if ρ defines a comodule. Hence the result follows. □

Proposition 4.1.12. *An affine group scheme is algebraic if and only if it has a faithful finite dimensional represenation. Furthermore, every affine group scheme may be obtained as an inverse limit of algebraic group schemes in which the projection maps are surjective.*

4.1.2 The category of represenations

Let G be an affine group scheme over k, $\mathrm{Rep}_k(G)$ the category of representations of G, and $\omega : \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$ be the forgetful functor. One may show that $\mathrm{Rep}_k(G)$ is a Tannakian category.

For R a k-algebra, define

$$\underline{\mathrm{Aut}}^{\otimes}(\omega)(R) := \mathrm{Isom}^{\otimes}(\phi_R \circ \omega, \phi_R \circ \omega),$$

where $phi_R : \mathrm{Vec}_k \to R - \mathrm{Mod}$ is given by $\phi_R(V) := V \otimes R$. An element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ consists of a family (λ_X) of R-linear automorphisms of $X \otimes R$ such that

1. $\lambda_{X\otimes Y} = \lambda_X \otimes \lambda_Y$,
2. $\lambda_k = \mathrm{id}_R$,
3. $\lambda_Y \circ (\alpha \otimes \mathrm{id}_R) = (\alpha \otimes \mathrm{id}_R) \circ \lambda_X$

for any morphism of G-representations $\alpha : X \to Y$.

Note that elements $g \in G(R)$ define elements of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$, and hence we have a map $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$.

Theorem 4.1.13. *The map $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is an isomorphism.*

Proof. We will only sketch the proof. Let $X \in \mathrm{Rep}_k(G)$, and let C_X be the strictly full subcategory of objects isomorphism to subquotients of objects of the form $P(X, X^\vee)$, where $P(x, y) \in \mathbb{N}[x, y]$, and

$$(\sum_{i,j} a_{i,j} x^i y^j)(X, X^\vee) := \bigoplus_{i,j} (X^{\otimes i} X^{\vee \otimes j})^{\oplus a_{i,j}}.$$

The map $\lambda \to \lambda_X$ identifies $\underline{\mathrm{Aut}}^{\otimes}(\omega|_{C_X})(R)$ with a subgroup of $GL(X \otimes R)$. Take G_X to be the image of G in $GL(X \otimes -)$. This defines a closed subgroup and

$$G_X(R) \subset \underline{\mathrm{Aut}}^{\otimes}(\omega|_{C_X})(R) \subset GL(X \otimes R).$$

If $V \in C_X$, and $t \in V$ is fixed by G, then

$$\begin{aligned} \alpha : k &\to V \\ a &\mapsto at \end{aligned}$$

is a G-equivariant map. Hence

$$\lambda_V(t \otimes 1) = (\alpha \otimes \mathrm{id})(\lambda_k(1)) = t \otimes 1$$

and hence $\underline{\mathrm{Aut}}^{\otimes}(\omega_{C_X})(R)$ i the subgroup of $GL(X \otimes R)$ fixing all tensors in representations of G_X fixed by G_X. By an 1982 result due to Deligne [**?**], this implies $G_X \cong \underline{\mathrm{Aut}}^{\otimes}(\omega_{C_X})$.

Next note that if $X' = X \oplus Y$, then $C_X \subset C_{X'}$, and the maps $G_{X'} \to G_X$ and $\underline{\mathrm{Aut}}^{\otimes}(\omega_{C_{X'}}) \to \underline{\mathrm{Aut}}^{\otimes}(\omega_{C_X})$ commute with the isomorphisms. Taking inverse limits, we find that $G \cong \underline{\mathrm{Aut}}^{\otimes}(\omega)$. $\square$

Lecture 5

5.1 Mixed Hodge structures

We now must define a mixed Hodge structure. In order to motivate this, let us briefly discuss how we will eventually define a motivic period. Recall that we defined (formal) periods as a tuple (X, Y, γ, ω), with $\gamma \in H_B^k(X,Y)^\vee$ and $\omega \in H_{dR}^k(X,Y)$. We will define, for an appropriate choice of category of motives, and a pair (ω_1, ω_2) of fiber functors, a motivic period as a triple (M, γ, η) for M a motive, $\gamma \in \omega_1(M)^\vee$, $\eta \in \omega_2(M)$. As cohomology should factor through the category of motives, we should be able to think of ω_1 as $H_B^\bullet$, and ω_2 as $H_{dR}^\bullet$. We will eventually take a simplified version of Delignes category of realisations as our category of motives. To be precise, we will take a motive as a triple (V_B, V_{dR}, c) consisting of two $\mathbb{Q}$-vector spaces and an isomorphism

$$c : V_{dR} \otimes \mathbb{C} \to V_B \otimes \mathbb{C}.$$

However, this is a bit too general to discuss periods in any real detail. For example, taking the cohomology of elliptic curves, we obtain isomorphic motives, despite different elliptic curves have different periods in general. As such, we need to impose some sort of conditions to make this triple somehow "geometric" in order to obtain a meaningful theory of periods.

On the Betti side, it will suffice to introduce a real Frobenius F_∞, arising from complex conjugation. On the deRham side, we need a mixed Hodge structure. This arises very naturally in geometry. Let M be a complex projective manifold of dimension n, and let $H^{p,q}(M) \subset H_d^{p+q}R(M, \mathbb{C})$ be spanned by classes locallly of the form

$$\sum_{\substack{I,J \subset \{1,\ldots,n\} \\ |I|=p,\ |J|=q}} f_{I,J}(z_1, \ldots, n) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

Theorem 5.1.1 (Hodge). *There is a canonical decomposition* $H^n(M, \mathbb{Q}) \otimes \mathbb{C} \cong \bigoplus_{p+q=n} H^{p,q}(M)$, *and* $H^{p,q}(M) \cong \overline{H^{q,p}(M)}$.

Furthermore, Hodge's theorem holds for any smooth projective variety over $\mathbb{Q}$. As such, we might be tempted to define a Hodge structure as a vector space with a decomposition as in Hodge's theorem, but the following definition is more convenient.

Definition 5.1.2. A pure Hodge structure over $\mathbb{Q}$ of weight n is a triple $(V_B, (V_{dR}, F^\bullet), c)$ where V_B, V_{dR} are $\mathbb{Q}$-vector spaces, $F^\bullet$ is an exhaustive decreasing filtration (called the Hodge filtration), and c is an isomorphism $V_{dR} \otimes \mathbb{C} \to V_B \otimes \mathbb{C}$ such that the induced filtration on $V_\mathbb{C} := V_B \otimes \mathbb{C}$ satisfies $V_\mathbb{C} = F^p V_\mathbb{C} \oplus \overline{F^{n+1-p} V_\mathbb{C}}$ for all p.

Remark 5.1.3. Note that by defining $F^pH^n(M,\mathbb{C}) := \oplus_{i\geq p}H^{i,n-i}$, we may construct a Hodge structure from the Hodge decomposition. We similarly get a Hodge decomposition from a Hodge structure by defining $V_{\mathbb{C}}^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^qV_{\mathbb{C}}}$.

Definition 5.1.4. A morphism of pure Hodge structures is a pair of $\mathbb{Q}$-linear maps (f_B, f_{dR}) : $(V_B, V_{dR}) \to (W_B, W_{dR})$ such that $f_{dr}(F^pV_{dr}) \subset F^pW_{dR}$, and $(f_B \otimes \mathrm{id}) \circ c_V = c_W \circ (f_{dR} \otimes \mathrm{id})$.

Remark 5.1.5. We leave it as an exercise to the reader to prove that every morphism of pure Hodge structures of different weights is zero.

Example 5.1.6. A Hodge-Tate structure of weight $-2n$ is defined to be

$$\mathbb{Q}(n) = (\mathbb{Q}, (\mathbb{Q}, F^\bullet), c_n)$$

where

$$\begin{aligned} F^{-n}\mathbb{Q} &:= \mathbb{Q} \\ F^{1-n}\mathbb{Q} &:= 0 \\ c_n(1 \otimes z) &= 1 \otimes \frac{z}{(2\pi i)^n} \end{aligned}$$

We call $\mathbb{Q}(1)$ the Tate structure, and $\mathbb{Q}(-1)$ is sometimes called the Lefschetz structure. As an exercise, we suggest the reader prove

$$\mathbb{Q}(n) \cong \mathbb{Q}(1)^{\otimes n} \text{ for } n \geq 0$$

and

$$\mathbb{Q}(-1) \cong \mathbb{Q}(1)^\vee = \mathrm{Hom}(\mathbb{Q}(1), \mathbb{Q}(0)).$$

Example 5.1.7. The cohomology $H^1(\mathbb{G}_m) = (H^1_B(\mathbb{G}_m), (H^1_{dR}(\mathbb{G}_m), F^\bullet), c)$ is isomorphic to $\mathbb{Q}(-1)$, despite $\mathbb{G}_m$ not being a smooth projective variety. The Hodge filtration is given by

$$\begin{aligned} F^1H^1_{dR}(\mathbb{G}_m) &= H^1_{dR}(\mathbb{G}_m) \\ F^2H^1_{dR}(\mathbb{G}_m) &= 0 \end{aligned}$$

and $c([\frac{dz}{z}]) = 2\pi i[\gamma_0]$. One method of proving this is to consider the Mayer-Vietoris sequence

$$H^1(\mathbb{A}^1) \oplus H^1(\mathbb{A}^1) \to H^1(\mathbb{G}_m) \to H^2(\mathbb{P}^1) \to H^2(\mathbb{A}^1) \oplus H^2(\mathbb{A}^1)$$

from which we can conclude $H^1(\mathbb{G}_m) \cong H^2(\mathbb{P}^1)$.

Remark 5.1.8. It will occasionally be useful to consider the Tate twist of a Hodge structure $H(n) := H \otimes \mathbb{Q}(n)$, i.e.

$$H(n) = (H_B, (H_{dR}, F^{\bullet+n}), (2\pi i)^{-n}c)$$

However, for general varieties, a pure Hodge structure is too restrictive. Instead, we can define a mixed Hodge structure.

Theorem 5.1.9 (Deligne)**.** *The (relative) cohomology of a quasiprojective vareity X has a mixed Hodge structure.*

Definition 5.1.10. A $\mathbb{Q}$-mixed Hodge structure (MHS) is a triple

$$V = ((V_B, W_\bullet^B), (V_{dR}, W_\bullet^{dR}, F^\bullet), c)$$

where V_B, V_{dR}, are Q-vector spaces; $W_\bullet^B$, $W_\bullet^{dR}$ are exhaustive increasing filtrations; $F^\bullet$ is an exhaustive, decreasing filtration; c is an isomorphism such that

$$c(W_n^{dR} V_{dR} \otimes \mathbb{C}) = W_n^B V_B \otimes \mathbb{C}$$

and

$$\mathrm{gr}_m^W V = (\mathrm{gr}_m^{W^B} V_B, (\mathrm{gr}_m^{W^{dR}} V_{dR}, F^\bullet), c)$$

is a pure Hodge structure of weight m.

Definition 5.1.11. A morphism of MHS is a pair of $\mathbb{Q}$-linear maps $(f_B, f_{dR}) : (V_B, V_{dR}) \to (U_B, U_{dR})$ such that f_B is a morphism of filtered vector spaces, f_{dR} is a morphism of bifiltered vector spaces, and $(f_B \otimes \mathrm{id}) \circ c_V = c_W \circ (f_{dR} \otimes \mathrm{id})$.

Theorem 5.1.12 (Deligne)**.** *The category of MHSs over $\mathbb{Q}$ is Tannakian with two fiber functors $\omega_B(V) = V_B$ and $\omega_{dR}(V) = V_{dR}$.*

Example 5.1.13. We call a MHS V a MHS of Tate type if $\mathrm{gr}_{2m+1}^W V = 0$ and $\mathrm{gr}_{2m}^W V \cong \mathbb{Q}(-m)^{\oplus n_m}$ for some non-negative integers n_m. Let $X = \mathbb{P} \setminus S$, for S a finite set of points. Then, similarly to $H^1(\mathbb{G}_m)$, we find $H^1(X) \cong \mathbb{Q}(-1)^{|S|-1}$.

Example 5.1.14. Let $X = \mathbb{G}_m$, $Y = \{1, 2\}$. We have a long exact sequence in relative cohomology

$$0 \to H^0(\{2\}) \to H^1(\mathbb{G}_m, \{1, 2\}) \to H^1(\mathbb{G}_m) \to 0.$$

As $H^0(\{2\}) \cong \mathbb{Q}(0)$, and $H^1(\mathbb{G}_m) \cong \mathbb{Q}(-1)$, we can construct a MHS on $H^1(\mathbb{G}_m, \{1, 2\})$ by demanding this be an exact sequence of MHS, which determines the filtrations on $V = H^1(\mathbb{G}_m, \{1, 2\})$. In particular

$$\begin{aligned} W_i^\bullet V_\bullet &= W_i^\bullet(\mathbb{Q}(0)_\bullet) \text{ for } i \leq 2 \\ W_2^\bullet V_\bullet &= V_\bullet \end{aligned}$$

and

$$\begin{aligned} F^0 V_{dR} &= V_{dR} \\ F^1 V_{dR} &= \mathbb{Q}(-1)_{dR} \\ F^2 V_{dR} &= 0 \end{aligned}$$

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