MMG710: Fourier Analysis - Exercise Session Problems LP1 2023, Gothenburg University

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The following is a collection of all the problems covered in exercise sessions as part of the course. Each session's problem set is separated by a short reminder on notation used in the following problems.

Notation

Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic, continuous, Riemann integrable function, and denote by $c_f : \mathbb{Z} \to \mathbb{C}$ its Fourier coefficients

$$c_f(n) := \int_{\frac{-1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} dx.$$

Problems

Problem 1. *Suppose we have a (uniformly convergent) series expansion*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}.$$

Write this as a trigonometric series

$$f(x) = \sum_{n\geq 0} A_n \cos(2\pi n x) + B \sin(2\pi n x).$$

Can we compute $\{A_n, B_n\}$ directly from f?

Problem 2. *Show the following implications:*

1. f *even* $\Rightarrow c_f(n) = c_f(-n) \Leftrightarrow B_f(n) = 0$,
2. f *odd* $\Rightarrow c_f(n) = -c_f(-n) \Leftrightarrow A_f(n) = 0$,
3. $f : \mathbb{R} \to \mathbb{R} \Rightarrow c_f(n) = \overline{c_f(-n)} \Leftrightarrow A_f(n),\ B_f(n) \in \mathbb{R}$,
4. $f : \mathbb{R} \to i\mathbb{R} \Rightarrow c_f(n) = -\overline{c_f(-n)} \Leftrightarrow A_f(n),\ B_f(n) \in i\mathbb{R}$.

In particular, note that if f is even and real valued, so is c_f, and if f is odd and real valued, c_f is odd and purely imaginary. If the Fourier series of f converges, the above implications become equivalences.

Problem 3. *1. Compute the infinite sum* $S = \sum_{m=1}^{\infty} \frac{1}{1+m^2}$*. Consider the Fourier series of the* 1*-periodic function* f *such that* $f(x) = \cosh(2\pi x)$ *for* $|x| \leq \frac{1}{2}$*.*

2. Compute $S_\alpha = \sum_{m=1}^{\infty} \frac{1}{\alpha^2+m^2}$ *for* $\alpha \in \mathbb{R}_{>0}$*.*

Problem 4. *Write* $f(x) = x$*,* $|x| \leq \frac{1}{2}$ *as a sine-series.*

Write $f(x) = |x|$ $|x| \leq \frac{1}{2}$ *as a cosine series. (Do not worry about proving convergence here. The goal is just to do the computation.)*

Problem 5. *Let* $0 < \alpha \leq 1$ *and suppose* $f : \mathbb{R} \to \mathbb{C}$ *is a continuous* 1*-periodic function such that*

$$|f(x) - f(y)| \leq C_\alpha |x - y|^\alpha$$

for some constant C_α *and all* $x, y \in \mathbb{R}$*.*

Prove

$$|c_f(n)| \leq \frac{2^{-(1+\alpha)} C_\alpha}{|m|^\alpha}.$$

(Hint: consider the Fourier coefficients of $g_m(x) := f(x) - f(x - \frac{1}{2m})$*.)*

Notation

Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic, continuous, Riemann integrable function, and denote by $c_f : \mathbb{Z} \to \mathbb{C}$ its Fourier coefficients

$$c_f(n) := \int_{\frac{-1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} dx.$$

Problems

Problem 6. *Let* $c_0, c_1, \ldots, c_n, \ldots$ *be a sequence of complex numbers and denote by* s_n *and* σ_n *the partial and Cesàro sums respectively:*

$$s_n := \sum_{k=0}^{n} c_k,$$

$$\sigma_n := \frac{1}{n} \sum_{k=0}^{n-1} s_k.$$

1. *Prove that if* $s_n \to s$*, then* $\sigma_n \to s$*,*
2. *, Is* $c_n = n$ *Cesàro summable? That is, does* $\lim_{n\to\infty} \sigma_n$ *exist?*
3. *Is* $c_n = (-1)^n n$ *Cesàro summable?*

Problem 7. *Suppose that* $\sum_{n\in\mathbb{Z}} |c_f(n)| \leq \infty$*. Show that*

$$\sum_{n\in\mathbb{Z}} |c_f(n)|^2 = \int_{\frac{-1}{2}}^{\frac{1}{2}} |f(x)|^2 dx.$$

Problem 8. *Using the above result, conclude* $\sum_{m=1}^{\infty}\frac{1}{m^4}=\frac{\pi^4}{90}$*. (Hint: recall the Fourier expansion of* x^2*)*

Problem 9 (2021, Exam 1, Q1)**.** *Which of the following hold for* $f(x)=x^3$*?*

1. $\sum_{m\in\mathbb{Z}} c_f(m)=0$,
2. $\sum_{m\in\mathbb{Z}} c_f(m)(-1)^m=\frac{1}{8}$,
3. $\sum_{m\in\mathbb{Z}} c_f(m)(-1)^m=\frac{-1}{8}$,
4. $\sum_{m\in\mathbb{Z}} c_f(m)^2=\frac{1}{2^6\cdot 7}$,
5. $\sum_{m\in\mathbb{Z}} c_f(m)^2=\frac{-1}{2^6\cdot 7}$.

Problem 10 (2020, Exam 1, Q8)**.** *Let* $f:\mathbb{R}\to\mathbb{C}$ *be a continuous* 1*-periodic function with absolutely summable Fourier coefficients. Show that*

$$|\frac{1}{n}\sum_{k=0}^{n-1} f(\frac{k}{n})-\int_0^1 f(x)dx|\leq\sum_{m\neq 0}|c_f(mn)|$$

for all $n\geq 1$*. (Hint: Consider* $A_n(m):=\frac{1}{n}\sum_{k=0}^{n-1}e^{\frac{2\pi ikm}{n}}$*. Show that* $A_n(m)$ *is non-zero if and only if* n *divides* m*.)*

Notation

Let $f:\mathbb{R}\to\mathbb{C}$ be a 1-periodic, continuous, Riemann integrable function, and denote by $c_f:\mathbb{Z}\to\mathbb{C}$ its Fourier coefficients

$$c_f(n):=\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi inx}dx.$$

For $f,g,h:\mathbb{R}\to\mathbb{C}$ continuous, Riemann integrable 1-periodic functions, define their (circular) *convolution* to be

$$(f*g)(x):=\int_{-\frac{1}{2}}^{\frac{1}{2}} f(y)g(x-y)dy.$$

Problems

Problem 11. *Denote by* $f:\mathbb{R}\to\mathbb{R}$ *the* 1*-periodic function*

$$f(x)=\sum_{n\in\mathbb{Z}}\frac{1}{1+(x+n)^2}$$

Compute the limit

$$I:=\lim_{q\to 0^+} q\cdot\sum_{n=1}^{\infty}|c_f(n)|^q$$

You may use without proof the relation

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}dx}{1+x^2} = \pi e^{-2\pi|\xi|}.$$

Problem 12. *Show the following properties of convolution (assume f, g, h are continuous):*

1. $f * g = g * f$,
2. $f * (g * h) = (f * g) * h$,
3. $c_{f*g}(n) = c_f(n)c_g(n)$

Convolution often makes functions "nicer". For example, $f * g$ is continuous for any Riemann integrable 1-periodic functions. This is easy to prove if f and g are continuous, but is not obvious in general.

We get a particularly nice property if f is continuously differentiable.

Problem 13. *Suppose f is a continuously differentiable 1-periodic function and g is a Riemann integrable 1-periodic function. Show that*

$$\frac{d}{dx}(f * g)(x) = (f' * g)(x).$$

Note that this implies that if f is smooth, then so if $f * g$ for any 1-periodic Riemann integrable g.

Problem 14. *Let $f, g : (-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ be given by*

$$f(x) = xe^{-x^2}$$
$$g(x) = \frac{1}{1+|x|^3}$$

and extend these 1-periodically to functions on $\mathbb{R}$. Which of the following are true?

1. $(f * g)(0) = 0$,
2. $(f * g)(0) = \sqrt{\pi}$,
3. $\int_{-\frac{1}{2}}^{\frac{1}{2}} (f * g)(x)dx = 0$.

When proving results about continuity of integrals of discontinuous functions, it may be helpful to approximate these functions by continuous counterparts. In the case of step functions, we often want to approximate them by *compactly supported* smooth functions, i.e. smooth functions $\phi : \mathbb{R} \to \mathbb{R}$ for which

$$\{x \in \mathbb{R} \mid \phi(x) \neq 0\}$$

is compact. Such approximations are called bump functions, and convolution with a bump function can provide a valuable smooth approximation to a non-smooth function.

Problem 15. *Prove that the function* $\phi : \mathbb{R} \to \mathbb{R}$

$$\phi(x) := \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

is a compactly supported smooth function.

Notation

Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic, continuous, Riemann integrable function, and denote by $c_f : \mathbb{Z} \to \mathbb{C}$ its Fourier coefficients

$$c_f(n) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} dx.$$

Problems

Problem 16. *Let* $a \in \mathbb{R} \setminus \mathbb{Z}$ *be a fixed real number. By considering the Fourier series of (the 1-periodic extension of)*

$$f(x) := \cos(2\pi a x) \quad x \in [-\frac{1}{2}, \frac{1}{2}]$$

show that

$$\pi \cot(\pi a) = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{n+a}.$$

Problem 17. *The Bessel functions* $J_n : \mathbb{R} \to \mathbb{R}$ *are defined by the relation*

$$e^{it \sin(2\pi x)} = \sum_{n \in \mathbb{Z}} J_n(t) e^{2\pi i n x}$$

for all $x \in \mathbb{R}$*. Compute*

$$I(t) := \sum_{n=-\infty}^{\infty} |J_n(t)|^2.$$

Problem 18. *Find a continuous* 1*-periodic solution to the ordinary differential equation*

$$f''(x) - f(x) = \frac{1}{1 - \frac{1}{2} e^{2\pi i x}}.$$

Problem 19. *Find Riemann integrable* $f(x)$ *on the interval* $[0, \frac{1}{2}]$ *such that the solution to the heat equation*

$$\partial_t u(x,t) = \partial_x^2 u(x,t) \quad x \in (0, \frac{1}{2}),\ t > 0$$

$$u(0,t) = u(\frac{1}{2}, t) = 0 \quad t \geq 0,$$

$$u(x,0) = f(x) \quad x \in [0, \frac{1}{2}],$$

satisfies $u(x, 3) = 5\sin(2\pi x) - \sin(14\pi x)$.

Problem 20. *Let* c, κ, L *be positive real numbers. Suppose we are given* $f, g : [0, L] \to \mathbb{R}$ *such that their odd* $2L$*-periodic extensions are twice continuously differentiable. Let* $u : [0, L] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *denote the solution to the following wave equation with damping*

$$\begin{aligned}
\partial_t^2 u(x,t) &= c^2 \partial_x^2 u(x,t) - 2\kappa \partial_t u(x,t) \quad x \in (0, L),\ t > 0,\\
u(0,t) &= u(L,t) = 0 \quad t \geq 0,\\
u(x,0) &= f(x) \quad x \in [0, L],\\
\partial_t u(x,0) &= g(x) \quad x \in [0, L].
\end{aligned}$$

Show that if $\kappa < \frac{c\pi}{L}$*, then* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly on* $[0, L]$.

Notation

Let V be a complex vector space equipped with an inner product

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}.$$

Denote by $\|v\|^2 := \langle v, v \rangle$ the associated norm. Recall that a set of elements $\{e_1, \ldots, e_n\}$ is called orthogonal if $\langle e_i, e_j \rangle = 0$ for all $i \neq j$. A set of elements $\{e_1, \ldots, e_n\}$ is called orthonormal if it is orthogonal and $\|e_i\| = 1$ for every i.

0.1 Problems

Problem 21. *Prove that for any* $x, y \in V$*, we have*

$$\langle x, y \rangle = \frac{1}{4}\left(\left(\|x+y\|^2 - \|x-y\|^2\right) + i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right.$$

Hence prove that the inner product on

$$\ell^2 := \{(a_n)_{n\geq 1} \mid \sum_{n\geq 1} |a_n|^2 \leq \infty\}$$

given by

$$\langle (a_n), (b_n) \rangle = \sum_{n\geq 1} a_n \overline{b}_n$$

always converges and is therefore well defined.

Problem 22. *Let* V *be the vector space of continuous functions on* $[-1, 1]$ *and define an inner product on* V *by*

$$\langle f, g \rangle := \int_{-1}^{1} f(x)\overline{g(x)}|x|^{\frac{1}{2}}\, dx.$$

Let $g_1(x) = 1$*,* $g_2(x) = x$*,* $g_3(x) = x^2 - \frac{3}{7}$*. Which of the following are true?*

1. g_1, g_2, g_3 *are orthogonal,*
2. g_1, g_2, g_3 *are orthonormal,*
3. $\|g_1 + g_2 + g_3\|^2 > \frac{3}{2}$,
4. $\|g_1 + g_2 + g_3\|^2 \leq \frac{3}{2}$
5. $\langle g_1, g_3 \rangle = 1$.

Problem 23. *Let V be a complex vector space with inner product $\langle \cdot, \cdot \rangle$. Suppose $(x_n)_{n\geq 1}$ is a sequence of elements of V and suppose there exists an element x such that*

$$\lim_{n\to\infty} \|x_n - x\| \to 0.$$

Show that $\lim_{n\to\infty} \|x_n\| = \|x\|$. Does the reverse implication hold?

Problem 24. *Show that*

$$\left(\sum_{n\geq 1} e^{-n} \frac{\sin(nx)}{n} \right)^2 \leq \frac{e^{-2}}{1 - e^{-2}} \sum_{n\geq 1} \frac{\sin^2(nx)}{n^2}$$

for every $x \in \mathbb{R}$. Does there exist any non-zero $x \in \mathbb{R}$ such that this is an equality?

Problem 25. *Which of the following inequalities hold?*

1. $|\int_0^1 \frac{\sin(2\pi x)}{\sqrt{1+x^2}} dx| \leq \frac{\sqrt{\pi}}{2\sqrt{2}}$,
2. $|\int_0^1 \frac{\sin(2\pi x)}{\sqrt{1+x^2}} dx| > \frac{\sqrt{\pi}}{2\sqrt{2}}$,
3. $\frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin\theta} d\theta \geq 1$ *for any* $x \neq 0$,
4. $\sum_{n\geq 1} \frac{\cos(n^3)}{2^n} \leq \left(\sum_{n\geq 1} \frac{\cos^2(n^3)}{2^n} \right)^{\frac{1}{2}}$,
5. $\sum_{n\geq 0} \frac{1}{2^n + 3^n} \geq 2$.

Notation

Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic, continuous, Riemann integrable function, and denote by $c_f : \mathbb{Z} \to \mathbb{C}$ its Fourier coefficients

$$c_f(n) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} dx.$$

Let V be a complex vector space equipped with an inner product

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}.$$

Denote by $\|v\|^2 := \langle v, v \rangle$ the associated norm. Recall that a set of elements $\{e_1, \ldots, e_n\}$ is called orthogonal if $\langle e_i, e_j \rangle = 0$ for all $i \neq j$. A set of elements $\{e_1, \ldots, e_n\}$ is called orthonormal if it is orthogonal and $\|e_i\| = 1$ for every i.

Problems

Problem 26. *Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic, twice continuously differentiable function. Show that*

$$\lim_{n \to \pm\infty} |n^2 c_f(n)| = 0.$$

Hint: Apply the Riemann Lebesgue Lemma.

Problem 27. *Show that if $f : \mathbb{R} \to \mathbb{C}$ is 1-periodic and continuously differentiable, the Fourier series of f converges absolutely. Hint: Apply Cauchy-Schwartz. Recall that $|c_{f'}(n) = 2\pi n|c_f(n)|$.*

Problem 28. *Let*

$$I = \int_0^1 \frac{\sin(\pi x)}{x^{\frac{1}{4}}} dx\,, \quad J = \int_1^\infty e^{-x^5}.$$

Which of the following are true?

1. $I < 1$,
2. $I \leq \frac{1}{3}$,
3. $I > \frac{1}{3}$,
4. $J \leq \frac{e^{-1}}{3\sqrt{3}}$,
5. $J > \frac{e^{-1}}{3\sqrt{3}}$

Hint: Is there a function $p(x)$ such that we can compute the integral of $p(x)e^{-x^5}$?

Problem 29. *On the interval $[-1, 1]$, define the Legendre polynomials by*

$$L_n(x) := \frac{d^n}{dx^2}(x^2 - 1)^n.$$

Consider the space of real-valued continuous functions on $[-1, 1]$ with inner product

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x)dx.$$

1. *If $f : [-1, 1] \to \mathbb{R}$ is infinitely differentiable, show that*
$$\langle L_n, f \rangle = (-1)^n \langle (x^2 - 1)^n, f^{(n)} \rangle.$$
2. *Prove that for $m \neq n$, $\langle L_n, L_m \rangle = 0$.*
3. *Hence show that any polynomial p of degree n such that $\langle p, x^k \rangle = 0$ for $k = 0, \ldots, n - 1$ is a multiple of L_n.*

Remark 1. *Any Riemann integrable function on $[-1, 1]$ can be well approximated by continuous functions. Any continuous function on $[-1, 1]$ can be well approximated by a polynomial. Combining these two claims, along with the best approximation theorem, imply that*

$$\lim_{N\to\infty} \|f - \sum_{n=0}^{N} \langle f, \mathcal{L}_n\rangle \mathcal{L}_n\| = 0$$

where $\mathcal{L}_n = L_n/\|L_n\|$. Thus, there exists a Legendre expansion of f which converges to f in a square-integrable sense.

Problem 30. *For $r \in [0, 1)$, define the Poisson kernel by*

$$P_r(x) := \sum_{n\in\mathbb{Z}} r^{|n|} e^{2\pi i n x},$$

where we take $r^0 = 1$ for all $r \in [0, 1)$. Show that

$$\lim_{r\to 1^-} \int_{-\frac{1}{2}}^{\frac{1}{2}} P_r(x - y) f(y) dy = f(x)$$

converges uniformly for all continuous 1-periodic functions $f : \mathbb{R} \to \mathbb{C}$. Hint: Consider the Key Lemma about Dirac families.

Notation

For $f : \mathbb{R} \to \mathbb{C}$ absolutely integrable, the Fourier transform of f is defined by

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

For f continuous with $\hat{f}$ absolutely integrable, the inverse Fourier transform recovers the original function

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Problems

Problem 31. *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function. Show the following*

- *If f is even, $\hat{f}$ is even,*
- *If f is odd, $\hat{f}$ is odd,*
- *If f is real valued, $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$,*
- *If f is totally imaginary, $\hat{f}(-\xi) = -\overline{\hat{f}(\xi)}$.*

In particular, note that if f is even and real, $\hat{f}$ is even and real. If the Fourier inversion theorem holds, then the prior implications are equivalences.

Problem 32. *Suppose f is absolutely integrable and continuously differentiable with absolutely integrable derivative. Show that*

$$\widehat{\frac{df}{dx}}(\xi) = 2\pi i\xi\hat{f}(\xi) \quad \text{and} \quad \frac{d\hat{f}}{d\xi}(\xi) = -2\pi i\widehat{xf}(\xi).$$

Problem 33. *Use the results of problem to to give a shorter proof of the result that*

$$f(x) = e^{-\pi x^2} \Rightarrow \hat{f}(\xi) = e^{-\pi\xi^2}.$$

Specifically, give a proof that avoids the auxiliary function ϕ used in the lectures. You may assume that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Problem 34. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable function with Fourier transform*

$$\hat{f}(\xi) = \begin{cases} 1 - |\xi| & |\xi| \leq 1, \\ 0 & |\xi| > 1. \end{cases}$$

Without computing the inverse Fourier transform (except possibly at $x = 0$) determine which of the following are true.

1. $\sum_{n\geq 1} f(n) = 0$,
2. $f(0) = 1$,
3. f *is even,*
4. f *is real valued.*

Problem 35. *Explicitly compute the inverse Fourier transform of*

$$\hat{f}(\xi) = \begin{cases} 1 - |\xi| & |\xi| \leq 1, \\ 0 & |\xi| > 1. \end{cases}$$

Hence show that

$$\frac{\pi^2}{\sin^2(\pi x)} = \sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^2}$$

for all $x \notin \mathbb{Z}$.

Hint: An antiderivative of $x\cos(ax)$ is $\frac{\cos(ax)}{a^2} + x\frac{\sin(ax)}{a}$.

Problem 36. *Suppose $u : \mathbb{R} \to \mathbb{C}$ is a infinitely differentiable function with absolutely integrable derivatives, solving the differential equation*

$$u'' - u = e^{-|t|}.$$

Determine the Fourier transform of u.

Notation

For $f : \mathbb{R} \to \mathbb{C}$ absolutely integrable, the Fourier transform of f is defined by

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

For f continuous with $\hat{f}$ absolutely integrable, the inverse Fourier transform recovers the original function

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

We denote by $u : \mathbb{R} \to \mathbb{R}$ the Heaviside step function

$$u(x) := \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Problems

Problem 37. *Let $f(x) = xe^x u(x)$. Which of the following equalities are true?*

1. $(f * f)(x) = x^2 e^{2x} u(x)$,
2. $(f * f)(x) = x^2 e^x u(x)$,
3. $(f * f)(x) = \frac{x^3}{6} e^x u(x)$,
4. $(f * f)(0) = 0$,
5. $(f * f)(0) = \int_0^\infty x^2 e^{2x} dx$.

Problem 38. *Let $\psi(x) = e^{i\pi x^2}$ and suppose $f : \mathbb{R} \to \mathbb{C}$ is a bounded continuous absolutely integrable function. SHow that*

$$\int_{-\infty}^{\infty} |(f * \psi)(x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Show that if $\phi(x) = e^{i\pi x}$, then

$$\int_{-\infty}^{\infty} |(f * \phi)(x)|^2 dx < \infty$$

if and only if $\hat{f}(\frac{1}{2}) = 0$.

Problem 39. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with constant $C > 0$ such that $|f(x)| \leq \frac{C}{1+x^2}$ for all $x \in \mathbb{R}$. Show the following:*

1. *Let $a \in \mathbb{R}$ and define $g_a(x) := f(x - a)$. Determine $\hat{g}_a(\xi)$.*
2. *By appropriate choice of a, show that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Hint: find a such that $\hat{f}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} (f(x) - g_a(x))\, e^{-2\pi i\xi x}dx$.

Problem 40. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be an absolutely integrable function such that* $\hat{f}(\xi) = \frac{1}{1+\xi^4}$. *Which of the following are true?*

1. $\widehat{f' * f'}(\xi) = \frac{-4\pi^2\xi^2}{(1+\xi^4)^2}$,
2. $\widehat{f' * f'}(\xi) = \frac{4\pi^2\xi^2}{(1+\xi^4)^2}$,
3. *If* $g(x) = f(x)\cos(2\pi x)$, $\hat{g}(\xi) = \frac{1}{2}\left(\frac{1}{1+(\xi+1)^4} + \frac{1}{1+(\xi-1)^4}\right)$,
4. *If* $g(x) = f(x)\cos(2\pi x)$, $\hat{g}(\xi) = \frac{1}{2}\left(\frac{1}{1+(\xi+2\pi)^4} + \frac{1}{1+(\xi-2\pi)^4}\right)$,
5. $f'(0) = \pi$.

Problem 41. *Let* $g(x) = e^{-x^4}$. *We will show that* $\hat{g}(\xi)$ *assumes negative values.*

1. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be an even absolutely integrable function. Show that if* $f(x) \geq 0$ *for all* $x \in \mathbb{R}$, *then*

$$4\hat{f}(\xi) \leq 3\hat{f}(0) + \hat{f}(2\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

   Hint: $3 - 4\cos\theta + \cos 2\theta = 8\sin^4\frac{\theta}{2}$

2. *Suppose* $\hat{g}(\xi) \geq 0$ *for all* ξ *and derive a contradiction by showing the inequality fails for values close to zero.*

   Hint: Recall $\hat{\hat{g}}(x) = g(-x)$*!*

Notation

For $f : \mathbb{R}^d \to \mathbb{C}$ absolutely integrable, the Fourier transform of f is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle\xi,x\rangle}dx$$

For f continuous with $\hat{f}$ absolutely integrable, the inverse Fourier transform recovers the original function

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i\langle\xi,x\rangle}d\xi.$$

Problems

Problem 42. *Prove the Heisenberg uncertainty principal in one dimension: let* $\psi : \mathbb{R} \to \mathbb{C}$ *be a smooth rapidly decaying function with absolutely integrable derivatives such that*

$$\int_{-\infty}^{\infty} |\psi(x)|^2dx = 1.$$

Prove that
$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\hat{\psi}(\xi)|^2d\xi\right) \geq \frac{1}{16\pi^2}.$$

Hint: Integrate $|\psi(x)|^2$ by parts and apply Cauchy-Schwartz

Remark 2. *In quantum mechanics, such a ψ is called a probability amplitude. The probability of a quantity associated to ψ lying in $[a,b]$ is given by*

$$\int_a^b |\psi(x)|^2dx$$

while the expected value and variance of measurements of that quantity are given by
$$\bar{x} := \int_{-\infty}^{\infty} x|\psi(x)|^2dx, \quad V_x := \int_{-\infty}^{\infty} (x-\bar{x})^2|\psi(x)|^2dx$$

respectively. In the case where $\psi(x)$ is associated to the position of a particle, $\hat{\psi}(\xi)$ corresponds to the momentum of that particle. Thus, we have shown that, for a particle with mean position and momentum 0,

$$V_{position}V_{momentum} \geq \frac{1}{16\pi^2}$$

By a change of variables, this result applies to particles with non-zero mean position and momentum as well.

Problem 43. *Let $f : \mathbb{R} \to \mathbb{C}$ be an infinitely differentiable function with Fourier transform $\hat{f}(\xi) = e^{-|\xi|^{\frac{1}{2}}}$. Which of the following are true?*

1. *f is even,*
2. *f is real valued,*
3. *$\int_0^\infty |f'(x)|^2dx = 30\pi^2$,*
4. *$f(0) = 2$,*
5. *$f'(0)$=1.*

Problem 44. *Suppose $f : \mathbb{R} \to \mathbb{C}$ has Fourier transform $\hat{f}(\xi) = \frac{1}{1+|\xi|^\pi}$. Show that*
$$f(0) > |f(x)|$$
for all $x \neq 0$.

Hint: Note that for every $x \in \mathbb{R}$ there exists $\theta_x \in \mathbb{R}$ such that $|f(x)| = e^{2\pi ix\theta_x}f(x)$. Apply Fourier inversion.

Problem 45. *Let $f : \mathbb{R}^2 \to \mathbb{C}$ be an absolutely integrable function with Fourier transform $\hat{f}(\xi_1, \xi_2) = \frac{1}{(1+\xi_1^2+\xi^2)^\alpha}$ for some $\alpha > 1$. Which of the following can we say about f?*

1. $f(0,0) = 0$,
2. $f : \mathbb{R}^2 \to \mathbb{R}$,
3. $\int_{\mathbb{R}^2} f(x,y)^2 dx\, dy = \frac{\pi}{2\alpha - 1}$,
4. $\int_{\mathbb{R}^2} f(x,y) dx\, dy = 1$,
5. $\sup_{(x,y)\in\mathbb{R}^2} |f(x,y)| > \frac{\pi}{2\alpha-1}$.

Problem 46. *Let A be a $d \times d$ positive definite symmetric real matrix and let $f_A : \mathbb{R}^d \to \mathbb{C}$ be the function*

$$f_A(x) = e^{-\pi\langle x, Ax\rangle}.$$

Compute $\hat{f}(0)$. Can you give an expression for $\hat{f}(\xi)$?

Hint: Recall that every positive definite symmetric real matrix can be written in the form $A = R^{-1}DR$ for a diagonal matrix D and a rotation matrix R. To compute $\hat{f}(\xi)$, it is sufficient to give an expression for $\hat{f}(R^{-1}\xi)$.

Notation

For $f : \mathbb{R}^d \to \mathbb{C}$ absolutely integrable, the Fourier transform of f is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x\rangle} dx$$

For f continuous with $\hat{f}$ absolutely integrable, the inverse Fourier transform recovers the original function

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i\langle \xi, x\rangle} d\xi.$$

Problems

Problem 47. *Suppose $f : \mathbb{R}^2 \to \mathbb{C}$ is doubly periodic:*

$$f(x+1, y) = f(x, y+1) = f(x,y)$$

for all $x, y \in \mathbb{R}$. Define the Fourier coefficients of f by

$$c_f(m,n) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x,y) e^{-2\pi i m x - 2\pi i n y} dx\, dy.$$

Suppose further that

$$\sum_{m,n\in\mathbb{Z}} |c_f(m,n)| < \infty.$$

and f is continuous. Define

$$f_n(x) = \sum_{m\in\mathbb{Z}} c_f(m,n)e^{2\pi imx}$$

and show that

$$f(x,y) = \sum_{n\in\mathbb{Z}} f_n(x)e^{2\pi iny}$$

for every $(x,y) \in \mathbb{R}$ and hence conclude that

$$f(x,y) = \sum_{m,n\in\mathbb{Z}} c_f(m,n)e^{2\pi imx+2\pi iny}$$

Problem 48. *Assume the two dimensional analogue of Parseval's theorem*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x,y)|^2 dx\, dy = \sum_{m,n\in\mathbb{Z}} |c_f(m,n|^2$$

for continuous doubly periodic $f : \mathbb{R}^2 \to \mathbb{C}$ with absolutely summable Fourier coefficients. Let $g : \mathbb{R}^2 \to \mathbb{C}$ be such that

$$|g(x,y)| \leq \frac{C}{1+x^2+y^2}$$

for all $(x,y) \in \mathbb{R}^2$. Suppose further that

$$|\hat{g}(\xi,\eta)| \leq \frac{C'}{1+\xi^2+\eta^2}$$

for all $(\xi,\eta) \in \mathbb{R}^2$. By considering

$$\phi_g(x,y) := \sum_{m,n\in\mathbb{Z}} g(x+m,y+n)$$

prove a two dimensional analogue of the Poisson summation formula.

Problem 49. *Determine an absolutely integrable function $f : \mathbb{R} \to \mathbb{C}$ such that*

$$e^{-x^4} = \int_{-\infty}^{\infty} f(x-y)e^{-|y|}dy$$

for all x.

Problem 50. *Let $u : \mathbb{R} \times [0,1] \to \mathbb{C}$ be a function twice continuously differentiable in each variable such that*

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0 \text{ for all } y \in (0,1)$$
$$u(x,0) = u(x,1) = \frac{1}{1+4\pi^2x^2} \int_{-\infty}^{\infty} |u(x,y)|dx < \infty \text{ for all } y \in (0,1)$$

Determine $u(x,y)$ as a convolution integral.

Hint: You may use that the Fourier transform of

$$\phi(x) = \frac{\sin(\pi a)}{2} \frac{1}{\cosh(\pi x) + \cos(\pi a)}$$

is equal to

$$\Phi(\xi) = \frac{\sinh(2\pi a\xi)}{\sinh(2\pi\xi)}$$

Problem 51. *Let $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a function twice continuously differentiable in each variable, such that*

$$\partial_t u(x,t) = 6\partial_x^2 u(x,t) + e^{-\pi x^2}$$
$$u(x,0) = e^{-|x|}$$
$$\int_{-\infty}^{\infty} |u(x,t)| dx < \infty \text{ for all } t > 0$$

Determine $u(x,t)$. You may express your answer as an inverse Fourier transform. For an extra challenge, express your answer in terms of the error function:

$$\mathrm{Erf}(x) = \int_0^x e^{-\pi y^2} dy$$

and a convolution integral. Feel free to try to evaluate the convolution integral in terms of the error function too!

Hint: $f(x) = x\,\mathrm{Erf}(x) - \frac{1}{2\pi} e^{-\pi x^2}$ is an antiderivative of $\mathrm{Erf}(x)$.

Notation

For $f : \mathbb{R} \to \mathbb{C}$ be (piecewise) continuous, right sided, a-exponentially integrable function. The Laplace transform is defined as

$$\mathcal{L}f(s) := \int_0^{\infty} f(x) e^{-sx} dx$$

and is well defined for all $s \in \mathbb{C}$ with sufficiently real part greater than a. If there exists $b \in \mathbb{R}$ such that $\mathcal{L}f(b + 2\pi i\xi)$ is absolutely integrable with respect to ξ, the inverse Laplace transform is given by

$$f(x) = \int_{-\infty}^{\infty} \mathcal{L}f(b + 2\pi i\xi) e^(b + 2\pi i\xi) x d\xi.$$

We often write $F(s)$ in place of $\mathcal{L}f(s)$ where it is unambiguous to do so.

If the order of exponential integrability is not specified, you map assume f is exponentially integrable for all $a > 0$.

We denote by $u(x)$ the Heaviside step function:

$$u(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Problems

Problem 52. *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous on $[0, \infty)$, right sided, and exponentially integrable. Show that*

1. *If $f : \mathbb{R} \to \mathbb{R}$, then the restriction of F to $\mathbb{R}$ is real valued,*
2. *If $f(x) = x^k g(x)$, then $\mathcal{L}f^l(s) = s^l F(s)$ for all $0 \leq l \leq k$,*
3. *If f is bounded, then*
$$\lim_{s\to\infty} sF(s) = \lim_{x\to 0^+} f(x),$$
4. *If $\lim_{x\to\infty} f(x) < \infty$, then*
$$\lim_{x\to\infty} f(x) = \lim_{s\to 0^+} sF(s),$$

Remark 3. *The final two properties hold more generally:*

$$\lim_{s\to\infty} sF(s) = \lim_{x\to 0^+} f(x),$$

always holds, and

$$\lim_{x\to\infty} f(x) = \lim_{s\to 0^+} sF(s),$$

holds if $sF(s)$ is finite for all s with $\Re(s) > 0$.

Problem 53. *Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are continuous on $[0, \infty)$, right sided, and exponentially integrable, satisfying the conditions for Laplace inversion. Show that*

$$\int_0^\infty F(x)g(x)dx = \int f(y)G(y)dy.$$

Problem 54. *Let $h(x) = xe^x u(x)$. Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ are right sided, and exponentially integrable functions, continuous on $[0, \infty)$, such that $f = h * g$. Which of the following are true?*

1. *If $g(x) = u(x)$, then $f(x) = (1 + xe^x + e^x)u(x)$,*
2. *If $g(x) = u(x)$, then $f(x) = (1 + xe^x - e^x)u(x)$,*
3. *If $f(x) = x^2 e^{2x} u(x)$, then $g(x) = (2 + 4x + x^2)e^{2x} u(x)$,*
4. *If $f(x) = x^2 e^{2x} u(x)$, then $g(x) = (2 + 4x + 2x^2)e^{2x} u(x)$,*
5. *If $g(x) = e^x u(x)$, then $f(x) = x^2 e^x u(x)$.*

Hint: The Laplace transform of $x^n e^{ax} u(x)$ is $\frac{n!}{(s-a)^{n+1}}$.

Problem 55. *Let $\alpha, \beta \in \mathbb{C}$ be complex numbers and denote by $h_{\alpha,\beta}$ the right sided, c-exponentially integrable, continuous on $[0, \infty)$ function with Laplace transform*

$$H_{\alpha,\beta}(s) = \frac{1}{(s-\alpha)(s-\beta)}$$

for $\Re(s) > c$, where $c = \max(\Re(\alpha), \Re(\beta))$. Which of the following are true?

- $h_{\alpha,\beta}(x) = \frac{1}{\beta-\alpha}\left(e^{\alpha x} - e^{\beta x}\right)u(x)$ *if* $\alpha \neq \beta$,
- $h_{\alpha,\beta}(x) = \frac{1}{\alpha-\beta}\left(e^{\alpha x} - e^{\beta x}\right)u(x)$ *if* $\alpha \neq \beta$,
- $h_{\alpha,\alpha}(x) = xe^{\alpha x}u(x)$,
- $h_{\alpha,\beta}(x)$ *is bounded if and only if* $\Re(\alpha) \leq 0$ *and* $\Re(\beta) \leq 0$
- $h_{\alpha,\alpha}(x) = \alpha x e^{\alpha x}$.

Hint: The Laplace transform of $x^n e^{ax}u(x)$ is $\frac{n!}{(s-a)^{n+1}}$.

Problem 56. *Let $f, h : \mathbb{R} \to \mathbb{C}$ be the right sided and 1-exponentially integrable functions where*

$$h(x) = \cosh(x)u(x) \quad \textit{and} \quad F(s) = \frac{e^{-s}}{s^4 - 1} \textit{ for } \Re(s) > 1.$$

Assume f is continuous on $[0, \infty)$. Find a right-sided and 1-exponentially integrable function $g : \mathbb{R} \to \mathbb{R}$ such that $f = h * g$. Give your answer as a real valued function. You may use without proof the following Laplace transforms

$$p(x) = e^{ax}u(x) \Rightarrow P(s) = \frac{1}{s-a}, \quad k(x) = \sin(x)u(x) \Rightarrow K(s) = \frac{1}{s^2+1}.$$

Problem 57. *Suppose $g(x)$ is right sided and exponentially integrable. Show that*

$$f(x) := \int_{-\infty}^{x} g(y)dy$$

is right sided. If $f(x)$ is also exponentially integrable, show $sF(s) = G(s)$.

For f, g related as above, find real valued g such that

$$(g * g)(x) - 2f(x) = \frac{x^3}{6} - x$$

for all $x > 0$. You may use the following Laplace transform

$$h(x) = \frac{x^n}{n!}u(x) \Rightarrow H(s) = \frac{1}{s^{n+1}}.$$

Notation

For $f : \mathbb{R} \to \mathbb{C}$ be (piecewise) continuous, right sided, a-exponentially integrable function. The Laplace transform is defined as

$$\mathcal{L}f(s) := \int_0^\infty f(x)e^{-sx}dx$$

and is well defined for all $s \in \mathbb{C}$ with sufficiently real part greater than a. If there exists $b \in \mathbb{R}$ such that $\mathcal{L}f(b + 2\pi i\xi)$ is absolutely integrable with respect to ξ, the inverse Laplace transform is given by

$$f(x) = \int_{-\infty}^\infty \mathcal{L}f(b + 2\pi i\xi)e^(b + 2\pi i\xi)xd\xi.$$

We often write $F(s)$ in place of $\mathcal{L}f(s)$ where it is unambiguous to do so.

If the order of exponential integrability is not specified, you map assume f is exponentially integrable for all $a > 0$.

We denote by $u(x)$ the Heaviside step function:

$$u(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Problems

Problem 58. *Let f be the right-sided and exponentially bounded function whose Laplace transform is given by $F(s) = (1 + s^2)^{-\frac{1}{2}}$ for $Re(s) > 0$. Let $0 \leq \alpha < \beta$. Find a right sided function y such that*

$$(y' * y')(x) + (y * y)(x) = \beta^2 (f * f)(x), \quad y(0) = \alpha,$$

for all $x > 0$.

Hint: You can use the Laplace transform pairs

$$g(x) = \cos(x)u(x) \leftrightarrow G(s) = \frac{s}{s^2 + 1}, \quad h(x) = \sin(t)u(t) \leftrightarrow H(s) = \frac{1}{s^2 + 1}, \quad Re(s) > 0.$$

Problem 59. *Let*

$$a(x) = e^x u(x), \quad b(x) = e^{-x} u(x), \quad c(x) = 2e^x u(x),$$

and determine an α-exponentially integrable, right sided solution to

$$(a * y')(x) - (b * y)(x) = c(x), \quad y(0) = 0.$$

Problem 60. *Find a right sided function $f : \mathbb{R} \to \mathbb{C}$ such that $f(0) = 0$ and*

$$(f' * f')(x) + (f * f)(x) = \int_0^x y \sin(x - y)dy, \quad x > 0.$$

Problem 61. *Let $\alpha, \beta \in \mathbb{R}$ and supposed that y is a right sided and exponentially integrable solution to*

$$(y' * y)(x) + \alpha y(x) = \beta x^2 u(x), \quad y(0) = \alpha.$$

Suppose y is continuous at $x = 0$ and $\beta > 0$. Determine α for which a solution exists and give one example of such a solution.

Problem 62. *Let $\lambda \in \mathbb{R}$ and denote by y_λ the unique right sided solution to the initial value problem*

$$y''_\lambda(x) + 2y'_\lambda(x) + \lambda y_\lambda(x) = u(x-2), \quad y(0)_\lambda = y'_\lambda(0) = 0.$$

For which λ is y_λ a bounded function on $[0, \infty)$.

Notation

Problems

Problem 63. *Let u denote the solution to the heat equation*

i) $\partial_t u(t,x) = 2\partial_x^2 u(t,x), \quad t > 0,\ x \in (0,1),$

ii) $u(t,0) = u(t,1) = 0, \quad t \geq 0,$

iii) $u(0,x) = x(1-x), \quad x \in [0,1].$

Compute $\lim_{t\to\infty} e^{2\pi^2 t} u(t,x)$. *You may use without proof that*

$$\int_0^1 x(1-x)\sin(\pi x)dx = \frac{4}{\pi^3}.$$

Problem 64. *Determine the solution u denote to the heat equation*

i) $\partial_t u(t,x) = \partial_x^2 u(t,x) + t\sin(\pi x), \quad t > 0,\ x \in (0,1),$

ii) $u(t,0) = u(t,1) = 0, \quad t \geq 0,$

iii) $u(0,x) = \sin(\pi x), \quad x \in [0,1].$

Problem 65. *Given $\lambda > 0$, let u_λ denote the solution to the heat equation*

i) $\partial_t u_\lambda(t,x) = \lambda\partial_x^2 u_\lambda(t,x), \quad t > 0,\ x \in (0,1),$

ii) $u_\lambda(t,0) = u_\lambda(t,1) = 0, \quad t \geq 0,$

iii) $u_\lambda(0,x) = \frac{2}{1+\cos^2(\pi x)} - 1, \quad x \in [0,1].$

Compute $\lim_{\lambda\to\infty} e^{\lambda\pi^2 t} u_\lambda(t,x)$.

Problem 66. *Find a function $u : [0,\infty) \times [0,1] \to \mathbb{R}$ such that*

i) $\partial_t^2 u(t,x) - \partial_t u(t,x) + \partial_x^2 u(t,x), \quad t > 0,\ x \in (0,1),$

ii) $u(t,0) = u(t,1) = 0, \quad t \geq 0,$

iii) $u(0,x) = \sin(2\pi x) - \sin(4\pi x), \quad x \in [0,1],$

iv) $\lim_{t\to\infty} u(t,x) = 0.$

Problem 67. *Given continuous* $f : [0, \frac{1}{2}] \to \mathbb{C}$ *whose odd extension to* $[-\frac{1}{2}, \frac{1}{2}]$ *is continuously differentiable on* $(-\frac{1}{2}, \frac{1}{2}]$*, let* $u_f : [0, \frac{1}{2}] \times [0, \infty) \to \mathbb{C}$ *be the unique solution to*

i) $\partial_t u_f(t,x) = \partial_x^2 u_f(t,x), \quad t > 0,\, x \in (0,1),$

ii) $u_f(t,0) = u_\lambda(t,1) = 0, \quad t \geq 0,$

iii) $u_f(0,x) = f(x), \quad x \in [0,1].$

Define the energy of u_f *as the integral*

$$E_f(t) := \int_0^{\frac{1}{2}} |u_f(x,t)|^2 dx$$

and show

$$E_f(t) \leq e^{-8\pi^2 t} \int_0^{\frac{1}{2}} |f(x)|^2 dx$$

It may be helpful to show $E_f'(t) \leq -8\pi^2 E_f(t)$ *and integrate* $E_f'(t)e^{8\pi^2 t}$ *by parts, or consider Parseval's theorem.*