MMG710: Fourier Analysis - LP1 2023, Gothenburg University

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Week 1

1 Motivating Fourier series

Fourier series a a method of representing a sufficiently nice function as a trigonometric series. Historically, this was motivated by the study of certain partial differential equations. For example, the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

for $u : [\frac{-1}{2}, \frac{1}{2}] \times \mathbb{R} \to \mathbb{R}$ with boundary conditions

$$u(\frac{-1}{2}, t) = u(\frac{1}{2}, t) = 0 \text{ for all } t \in \mathbb{R}$$

may be solved as follows. Consider solutions of the form $u(x,t) = X(x)T(t)$. For non-zero solutions $u(x,t)$, the wave equation may be rewritten as

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

Since the left hand side is only a function of x and the right hand side is only a function of t, they must both be equal to some real constant λ. Hence

$$X''(x) = \lambda X(x).$$

If $\lambda = 0$, $X(x) = Ax + B$ for some constants A, B. The boundary constraints then imply that $A = B = 0$, so we get the trivial solution.

Similarly, if $\lambda > 0$, we obtain $X(x)$ is a sum of real exponentials, and hence there are no non-trivial solutions with our boundary conditions. Thus, we must have $\lambda = -c^2$ for some real c. Assuming this, we find

$$X(x) = A\cos(cx) + B\sin(cx).$$

Considering again our boundary constraints, we find we must have $c = 2\pi n$ for some $n \in \mathbb{Z}$. Without loss of generality we can take $n \geq 0$. We must also have that $B = 0$.

Thus, a general solution to the wave equation will be of the form

$$u(x,t) = \sum_{n\geq 0} T_n(t)\cos(2\pi nx)$$

for some functions $T_n(t)$. If we have an initial condition

$$u(x,0) = \sum_{n\geq 0} T_n(0)\cos(2\pi nx) = f(x)$$

we can solve this differential equation only if we can find constants $A_n = T_n(0)$ such that

$$\sum_{n\geq 0} A_n \cos(2\pi nx) = f(x).$$

The goal for this week to to describe how to compute these constants and discuss for what f such a series expansion can exist. More generally, we will consider when a series expansion of the form

$$f(x) = A_0 + \sum_{n\geq 1} A_n\cos(2\pi nx) + B_n \sin(2\pi nx)$$

or equivalently

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$$

exists.

2 Computing Fourier coefficients

Definition 1. *Let f be a Riemann integrable function that is 1-periodic on $\mathbb{R}$ or defined on $[-\frac{1}{2},\frac{1}{2}]$. Define its Fourier coefficients $c_f : \mathbb{Z} \to \mathbb{C}$ by*

$$c_f(n) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi inx}dx.$$

Lemma 1. *If we have a (uniformly convergent) series expansion*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx},$$

then $c_n = c_f(n)$ for all n.

Proof.

$$\begin{aligned} c_f(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi inx}dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m\in\mathbb{Z}} c_n e^{2\pi i(m-n)x}dx \end{aligned}$$

By the dominated convergence theorem, we can swap the order of integration and summation and hence

$$\begin{aligned} c_f(n) &= \sum_{m\in\mathbb{Z}} c_m \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(m-n)x} dx \\ &= \sum_{m\in\mathbb{Z}} c_m \delta_{m,n} \\ &= c_n, \end{aligned}$$

where

$$\delta_{m,n} = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ otherwise.} \end{cases}$$

□

Thus, to every Riemann integrable function $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$, we can associate a Fourier series

$$\sum_{n\in\mathbb{Z}} c_f(n) e^{2\pi i n x}$$

that is the unique possible exponential series expansion. However, it is not guaranteed that such a series converges at all, let alone to $f(x)$. For sufficiently nice functions, we will find that this is the case, but:

1. Near discontinuities, the Fourier series (if it converges) converges to a local average rather than the value of f
2. The Fourier series depends only on the integral of f and so we can change f on a set of measure 0 without changing the series
3. There are continuous functions for which the Fourier series does not converge at all at certain points

Nevertheless, for continuous functions, we can guarantee a certain type of convergence, and the Fourier coefficients still encode all the important information.

Proposition 1. *Let $f, g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ be Riemann integrable functions that are continuous at a point $x_0 \in (-\frac{1}{2}, \frac{1}{2})$. Then if $c_f(n) = c_g(n)$ for all n, $f(x_0) = g(x_0)$, i.e. a function is determined by its Fourier coefficients where it is continuous*

Proof. We will consider only the case where f and g are real valued. The complex case follows by considering real and imaginary parts.

By extending f, g to 1-periodic functions and translating if necessary, we can assume x_0=0. By considering $h = f - g$, it suffices to show that, if h is continuous at 0 and $c_h(n) = 0$ for all n, then $h(0) = 0$. Suppose otherwise, without loss of generality, that $h(0) > 0$.

Since h is continuous at 0, there exist $\delta > 0$ such that

$$h(x) > \frac{h(0)}{2}$$

for all $|x| < \delta$. Let

$$p(x) = \varepsilon + \frac{1}{2}\left(e^{2\pi ix} + e^{-2\pi ix}\right) = \varepsilon + \cos(2\pi x)$$

and denote by $p_k(x) := p(x)^k$. We choose ε sufficiently small such that $|p(x)| < 1 - \frac{\varepsilon}{2}$ for $|x| \geq \delta$ and choose $0 < \eta < \delta$ such that $p(x) > 1 + \frac{\varepsilon}{2}$ for $|x| < \eta$.

Then, $\int_{-\frac{1}{2}}^{\frac{1}{2}} h(x)p_k(x)dx$ is a linear combination of Fourier coefficients (as $p_k(x)$ is a linear combination of $e^{2\pi inx}$) and hence is equal to 0 for all k.

As h is continuous on a compact interval, there exists some $M > 0$ such that $|h(x)| < M$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$. We then note that

$$|\int_{\delta \leq |x| \leq \frac{1}{2}} h(x)p_k(x)dx| \leq M\left(1 - \frac{\varepsilon}{2}\right)^k, \quad \text{which tends to 0,}$$

$$\int_{\eta \leq |x| \leq \delta} h(x)p_k(x)dx \geq 0,$$

$$\int_{|x| \leq \eta} h(x)p_k(x)dx \geq \frac{\eta h(0)}{2}(1 + \frac{\varepsilon}{2})^k \to_{k\to\infty} \infty.$$

Thus, the integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} h(x)p_k(x) = 0$$

is a sum of a positive number, plus something small, plus something that tends to infinity for large k, and hence tends to infinity as $k \to \infty$. But this is a contradiction. Hence $h(0) = 0$. $\square$

Corollary 1. *If $f, g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ are continuous and $c_f(n) = c_g(n)$ for all n, then $f = g$.*

Corollary 2. *If $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ is continuous and $\sum_{n\in\mathbb{Z}} c_f(n)e^{2\pi inx}$ converges to a continuous function, then*

$$f(x) = \sum_{n\in\mathbb{Z}} c_f(n)e^{2\pi inx}.$$

In particular, if $\sum_{n\in\mathbb{Z}} |c_f(n)| < \infty$, then

$$f(x) = \sum_{n\in\mathbb{Z}} c_f(n)e^{2\pi inx}.$$

Proof. The first statement is an immediate application of the previous corollary. The second follows from the Weierstrauss M-test: if the sum converges absolutely, the series converges uniformly and hence to a continuous function. $\square$

Exercise 1. *By considering the Fourier coefficients of the function* $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$*,* $f(x) = x^2$*, compute the sums*

$$\sum_{m>0} \frac{1}{m^2} \qquad \sum_{m>0} \frac{(-1)^m}{m^2}.$$

While this is an important sufficient condition, checking absolute convergence of the sum of the Fourier coefficients can be tedious, and requires that we determine the Fourier series before knowing whether it is meaningful. For sufficiently differentiable functions, we can confirm convergence in advance.

Proposition 2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a twice continuously differentiable* 1*-periodic function. Then there exists a constant* B *such that* $|c_f(n)| \leq \frac{B}{n^2}$ *for all* $n \neq 0$*. In particular,* $\sum_{n\in\mathbb{Z}} |c_f(n)| < \infty$*, and so the Fourier series of* f *converges uniformly to* f*.*

Proof. Integrating by parts and using the 1-periodicity of $f(x)$ and $f'(x)$, we get

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi inx}dx =& \frac{1}{2\pi in} \int_{-\frac{1}{2}}^{\frac{1}{2}} f'(x)e^{-2\pi inx}dx \\ =& \frac{-1}{4\pi^2 n^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} f''(x)e^{-2\pi inx}dx. \end{aligned}$$

As $f''(x)$ is continuous, it is bounded on $[-\frac{1}{2}, \frac{1}{2}]$ by some constant M, and hence

$$\begin{aligned} |c_f(n)| =& \mid \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi inx}dx \mid \\ \leq& \frac{M}{4\pi^2 m^2}. \end{aligned}$$

□

3 Cèsaro summation and Dirac families

While, in general, continuity of $f(x)$ is not sufficient to guarantee the convergence of the partial sums

$$(S_N f)(x) := \sum_{n=-N}^{N} c_f(n)e^{2\pi inx},$$

it is enough to guarantee the convergence in a different sense.

Definition 2. *Given a sequence of complex numbers* $c_0, c_1, \ldots$*, the* N^{th} *partial sum is*

$$s_N := \sum_{n=0}^{N} c_n.$$

The N^{th} Cèsaro sum is defined to be

$$\sigma_N := \frac{1}{N}\sum_{n=0}^{N-1} s_n.$$

Cèsaro summation gives a way of sometimes assigning a meaninful value to a not-necessarily-convergent series.

Example 1. *Consider the sequence $c_n = (-1)^n$. The partial sums*

$$s_N = \begin{cases} 1 \text{ if } N \text{ is even,} \\ 0 \text{ if } N \text{ if odd,} \end{cases}$$

do not converge. However

$$\sigma_N = \begin{cases} \frac{1}{2} \text{ if } N \text{ is even,} \\ \frac{1}{2}(1+\frac{1}{N}) \text{ if } N \text{ is odd,} \end{cases}$$

converges to $\frac{1}{2}$, the "average" values of the series.

Exercise 2. *Suppose we have a sequence $\{c_n\}_{n\geq 0}$ such that $s_N \to s$ converges to a finite limit. Show that $\sigma_N \to s$.*

We claim that the Cèsaro sum of the Fourier series of a continuous function $f(x)$ converges to $f(x)$. In the language of Fourier series, the N^{th} Cèsaro sum is usually referred to as the N^{th} Fejér sum.

Definition 3. *For a Riemann integrable function 1-periodic function f, define its N^{th} Fejér sum to be*

$$f_N(x) := \sum_{|n|\leq N}\left(1-\frac{|n|}{N}\right)c_f(n)e^{2\pi inx}$$

for all $N \geq 0$.

The N^{th} Fejér kernel is defined by

$$F_N(x) := \sum_{|n|\leq N}\left(1-\frac{|n|}{N}\right)e^{2\pi inx}.$$

Lemma 2.

- *For $f : \mathbb{R} \to \mathbb{C}$ a Riemann integrable 1-periodic function*

$$f_N(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(x-y)f(y)dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(y)f(x-y)dy,$$

- $$F_N(x) = \begin{cases} \frac{1}{N}\left(\frac{\sin(\pi Nx)}{\sin(\pi x)}\right)^2 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

  ,

- *If* $(S_N f)(x) := \sum_{n=-N}^{N} c_f(n) e^{2\pi i n x}$*, then*

$$f_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} (S_n f)(x),$$

which is to say that the Fejér sums are the Cèsaro sums associated to the Fourier series of f

Exercise 3. *Prove the above lemma.*

Our goals for the next section is to prove Fejér's theorem.

Theorem 1. *Let* $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ *be a continuous function. Then* $f_N(x)$ *converges uniformly to* $f(x)$*.*

We will develop a slightly more general theory in order to prove this.

Definition 4. *Fix a real* $R > 0$ *and let* $(\rho_N)_{N \geq 1}$ *be a sequence of non-negative even functions on* $[-R, R]$ *such that*

$$\frac{1}{2R} \int_{-R}^{R} \rho_N(y) dy = 1$$

for all N*. We call* (ρ_N) *a Dirac family if for every* $0 < \delta < R$

$$\lim_{N \to \infty} \int_{\delta \leq |x| \leq R} \rho_N(y) dy = 0.$$

Remark 1. *We can weaken the non-negativity requirement in the definition of a Dirac family by replacing it with the condition that there exist* $M > 0$ *such that*

$$\int_{-R}^{R} |\rho_N(y)| dy < M$$

for all N*. We will not consider any such families, but all results we show can be extended to such families.*

Example 2. 1. *For* $R = 2$*, and* $\rho_N = 2N\chi_{[-\frac{1}{N}, \frac{1}{N}]}$*, where*

$$\chi_I(x) := \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\int_{\delta < |x| \leq \frac{1}{2}} \rho_N(y) dy = 0$$

for all $N > \frac{1}{\delta}$*, from which it is clear this is a Dirac family.*

2. *The Fejér kernels* (F_N) *are a Dirac family for* $R = \frac{1}{2}$*. Note that*

$$\int_{\delta \leq |x| \leq \frac{1}{2}} F_N(y) dy \leq \frac{1}{N} \frac{1}{(\sin(\pi\delta))^2} \to 0.$$

3. *For certain constants* $\ell_N \in \mathbb{R}$*, the collection* $\left(L_N(x) = \ell_N(1-x^2)^N\right)$ *is a Dirac family for* $R = 1$*.*

Exercise 4. *Verify the above examples are Dirac families. Determine* ℓ_N*. A helpful inequality for* (L_N) *is that there exist positive constants* C_1, C_2 *such that*

$$C_1 n^n e^{-n} \leq n! \leq C_2 (n+1)^{n+1} e^{-n}$$

for all $n \geq 0$*.*

Note that the Dirichlet kernel

$$D_N = \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$$

satisfies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(y) f(x-y) dy = (S_N f)(x)$$

but is *not* an example of a Dirac family. It is not non-negative and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mid D_N(y) \mid dy$$

grows like $\log(N)$, so doesn't satisfy even the more general conditions. As such, we cannot apply the following lemma to it.

Lemma 3. *Let* (ρ_N) *be a Dirac family on* $[-R, R]$ *and let* $f : [-2R, 2R] \to \mathbb{C}$ *be Riemann integrable.*

1. *If* $x \in [-R, R]$ *has that*

$$f(x^-) := \lim_{y \to x^-} f(y),$$
$$f(x^+) := \lim_{y \to x^+} f(y)$$

   both exist, then

$$\lim_{N\to\infty} \frac{1}{2R} \int_{-R}^{R} \rho_N(y) f(x-y) dy = \frac{1}{2}\left(f(x^-) + f(x^+)\right).$$

   In particular, if f *is continuous at* x*, then*

$$\lim_{N\to\infty} \frac{1}{2R} \int_{-R}^{R} \rho_N(y) f(x-y) dy = f(x).$$

2. *If* f *is continuous on* $[-2R, 2R]$*, then*

$$\lim_{N\to\infty} \frac{1}{2R} \int_{-R}^{R} \rho_N(y) f(x-y) dy$$

   converges uniformly to $f(x)$ *on* $[-R, R]$*.*

Proof. In order to prove the first step, consider $x \in [-R, R]$ such that $f(x^-)$ and $f(x^+)$ exist. Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\mid f(x^-) - f(x-y) \mid < \varepsilon \text{ for all } 0 \leq y < \delta \\ &\mid f(x^+) - f(x-y) \mid < \varepsilon \text{ for all } -\delta < y \leq 0. \end{aligned}$$

Note that, since $\rho_N(y)$ is even

$$\frac{1}{2R}\int_{-R}^{0} \rho_N(y)dy = \frac{1}{2R}\int_{0}^{R} \rho_N(y)dy = \frac{1}{2}$$

for all $N > 0$. We then consider

$$\begin{aligned} \frac{1}{2R}\int_{-R}^{R} \rho_N(y) f(x-y)dy - \frac{1}{2}\left(f(x^-) + f(x^+)\right) = &\frac{1}{2R}\int_{-R}^{0} \rho_N(y)\left(f(x-y) - f(x^-)\right)dy \\ &+ \frac{1}{2R}\int_{0}^{R} \rho_N(y)\left(f(x-y) - f(x^+)\right)dy \end{aligned}$$

where we have used the above integral representation of $\frac{1}{2}$. Thus

$$\begin{aligned} \mid \frac{1}{2R}\int_{-R}^{R} \rho_N(y) f(x-y)dy - \frac{1}{2}\left(f(x^-) + f(x^+)\right) \mid = &\frac{1}{2R}\int_{-R}^{0} \rho_N(y) \mid f(x-y) - f(x^-) \mid dy \\ &+ \frac{1}{2R}\int_{0}^{R} \rho_N(y) \mid f(x-y) - f(x^+) \mid dy \end{aligned}$$

As f is continuous on a compact interval, it is bounded, and so we have

$$\begin{aligned} \mid f(x-y) - f(x^+) \mid &\leq \mid f(x-y) \mid + \mid f(x^+) \mid \\ &\leq 2 \sup_{x \in [-2R, 2R]} |f(x)| =: 2||f||_\infty. \end{aligned}$$

Therefore

$$\frac{1}{2R}\int_{\delta}^{R} \rho_N(y) \mid f(x-y) - f(x^+) \mid dy \leq \frac{||f||_\infty}{R} \rho_N(y)dy.$$

Since (ρ_N) is a Dirac family, there exists N_0 such that for all $N > N_0$, this is bounded by ε.

By construction

$$\frac{1}{2R}\int_{0}^{\delta} \rho_N(y) \mid f(x-y) - f(x^+) \mid dy \leq \frac{\varepsilon}{2R}\int_{0}^{\delta} \rho_N(y)dy,$$

which is bounded above by ε.

Thus

$$\frac{1}{2R}\int_{0}^{R} \rho_N(y) \mid f(x-y) - f(x^+) \mid dy$$

is bounded above by 2ε. We may similarly bound

$$\frac{1}{2R}\int_{-R}^{0} \rho_N(y) \mid f(x-y) - f(x^-) \mid dy$$

and hence

$$\frac{1}{2R} \mid \int_{-R}^{R} \rho_N(y) f(x-y) dy - \frac{1}{2}\left(f(x^-) + f(x^+)\right) < 4\varepsilon$$

for all N larger than some N_0. Therefore

$$\lim_{N\to\infty} \frac{1}{2R}\int_{-R}^{R} \rho_N(y) f(x-y) dy = \frac{1}{2}\left(f(x^-) + f(x^+)\right).$$

In order to prove the second statement, we largely recreate the above proof. Since f is continuous on $[-2R, 2R]$, it is uniformly continuous and so for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\mid f(x-y) - f(y) \mid < \epsilon$$

for all $|y| < \delta$ and all $x \in [-R, R]$ (or any interval strictly contained within $[-2R, 2R]$.

We can write

$$\begin{aligned}\frac{1}{2R}\int_{-R}^{R} \rho_N(y) f(x-y) dy - f(x) =& \frac{1}{2R}\int_{-R}^{R} \rho_N(y)\left(f(x-y) - f(x)\right) dy \\ =& \frac{1}{2R}\int_{\delta\le|y|\le R} \rho_N(y)\left(f(x-y) - f(x)\right) dy \\ &+ \frac{1}{2R}\int_{|y|\le\delta} \rho_N(y)\left(f(x-y) - f(x)\right) dy\end{aligned}$$

Similarly to above, we must therefore have

$$\frac{1}{2R}\int_{\delta\le|y|\le R} \rho_N(y)|f(x-y) - f(x)| dy \le \frac{||f||_\infty}{R}\int_{\delta\le|y|\le R} \rho_N(y) dy$$

which we can bound above by ε for N sufficiently large. We also have

$$\frac{1}{2R}\int_{|y|\le\delta} \rho_N(y)|(f(x-y) - f(x)| dy \le \frac{\varepsilon}{2R}\int_{|y|\le\delta} \rho_N(y) dy$$

is bounded above by ε for all N. Hence, for sufficiently large N

$$\mid \frac{1}{2R}\int_{-R}^{R} \rho_N(y) f(x-y) dy - f(x) \mid < 2\varepsilon$$

for all $x \in [-R, R]$. Therefore

$$\frac{1}{2R}\int_{-R}^{R} \rho_N(y) f(x-y) dy$$

converges uniformly to $f(x)$. □

Corollary 3 (Fejér's Theorem)**.** *For f a continuous 1-periodic function, $f_N \to f$ uniformly*

Proof. This follows from applying the above lemma with $\rho_N = F_N$. $\square$

Corollary 4. *Trigonometric polynomials are dense in the space of continuous 1-periodic functions with respect to the $|| - ||_\infty$ norm, i.e. for a given continuous 1-periodic function f and every $\varepsilon > 0$, there exists a Laurent polynomial $\sum_{n=-N}^{N} a_n e^{2\pi i n x}$ in $e^{2\pi i x}$ such that*

$$|| \sum_{n=-N}^{N} a_n e^{2\pi i n x} - f(x) ||_\infty < \varepsilon$$

Proof. Take

$$\sum_{n=-N}^{N} a_n e^{2\pi i n x} = f_N(x)$$

for N sufficiently large. $\square$

Week 2

4 Hardy's Tauberian Theorem

Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic, Riemann integrable function, and denote by $c_f : \mathbb{Z} \to \mathbb{C}$ its Fourier coefficients

$$c_f(n) := \int_{\frac{-1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi i n x} dx.$$

Thus far, we have shown that for f continuous

$$\sum_{n\in\mathbb{Z}} |c_f(n)| < \infty \quad \Rightarrow \quad \sum_{n\in\mathbb{Z}} c_f(n)e^{2\pi i n x} = f(x)$$

and it converges uniformly. A sufficient condition for this is for f to be twice continuously differentiable. We also have Fejér's Theorem, which guarantees convergence of the Fejér sums

$$\lim_{N\to\infty} \sum_{|n|\leq N} \left(1 - \frac{|n|}{N}\right) c_f(n)e^{2\pi i n x} = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

However, neither of these apply to the Fourier series of something like $f(x) = x$ for $x \in (-\frac{1}{2}, \frac{1}{2}]$, or a square wave

$$f(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \frac{1}{2}, \\ -1 \text{ if } -\frac{1}{2} < x < 0, \end{cases}$$

both of which have Fourier coefficients that decay like $\frac{1}{n}$. Nevertheless, numerically the partial sums of their Fourier series seem to converge to $\frac{1}{2}\left(f(x^+) + f(x^-)\right)$. We will show that this is indeed always the case. In fact, we will prove something a bit more general.

Theorem 2 (Hardy's Tauberian Theorem)**.** *Let $c_0, c_1, \ldots \in \mathbb{C}$ be a sequence of complex numbers and let $s_N = \sum_{n=0}^{N} c_n$. Suppose that the Cèsaro limit exists*

$$s = \lim_{N\to\infty} \frac{1}{N}(s_0 + \cdots + s_{N-1})$$

and that there exists a constant $C > 0$ such that $n|c_n| \leq C$ for all $n > 0$. Then $\lim_{N\to\infty} s_N$ exists and is equal to s.

Corollary 5. *Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic Riemann integrable function. Suppose that there exists a constant $C > 0$ such that $|nc_f(n)| \leq C$ for all $n \neq 0$. Then, at every x for which the one sided limits $f(x^+)$ and $f(x^-)$ exist, we have*

$$lim_{N\to\infty} \sum_{n=-N}^{N} c_f(n)e^{2\pi i n x} = \frac{1}{2}\left(f(x^+) + f(x^-)\right).$$

Proof. Apply Hardy's Tauberian theorem to $c_0 = c_f(0)$, $c_n = c_f(n)e^{2\pi inx} + c_f(-n)e^{-2\pi inx}$. From our Key Lemma last week, specifically the case of Fejér's theorem, the Fejer sums of f converge to

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

and the N^{th} Fejer sum is equal to

$$\sum_{|n|\leq N}\left(1 - \frac{|n|}{N}\right)c_f(n)e^{2\pi inx} = \frac{1}{N}\sum_{n=0}^{N-1}(S_n f)(x).$$

□

Remark 2. *Note that, unlike our Key Lemma or Fejer's theorem, this corollary only tells us about pointwise convergence, not uniform convergence.*

Proof. Let $c_0, c_1, \ldots \in \mathbb{C}$ be a sequence of complex numbers and define

$$\begin{aligned} s_N &:= c_0 + c_1 + \cdots + c_N, \\ \sigma_N &:= \frac{1}{N}\sum_{n=0}^{N-1} s_n, \\ \kappa_N &:= N\sigma_N = \sum_{n=0}^{N-1}(N-n)c_n. \end{aligned}$$

We assume $s = \lim_{N\to\infty}\sigma_N$ exists. Note that

$$\begin{aligned} \kappa_{N+\ell} - \kappa_N &= \sum_{n=0}^{N+\ell-1}(N+\ell-n)c_n - \sum_{n=0}^{N-1}(N-n)c_n \\ &= \sum_{n=0}^{N-1}(N+\ell-n)c_n + \sum_{n=N}^{N+\ell-1}(N+\ell-n)c_n - \sum_{n=0}^{N-1}(N-n)c_n \\ &= \ell\sum_{n=0}^{N-1}c_n + \sum_{n=0}^{\ell-1}(\ell-n)c_{N+n} \\ &= \ell s_{N-1} + \mathcal{S}_{N,\ell} \end{aligned}$$

for every $\ell \geq 1$. Hence

$$s_{N-1} = \frac{1}{\ell}\left(\kappa_{N+\ell} - \kappa_N\right) - \frac{\mathcal{S}_{N,\ell}}{\ell}.$$

Since we have assumed there exists $C > 0$ such that $n|c_n| \leq C$ for all $n > 0$, we have that

$$|S_{N,\ell}| \leq \sum_{n=0}^{\ell-1}(\ell-n)|c_{N+\ell}| \leq \sum_{n=0}^{\ell-1}\frac{\ell C}{N+k} \leq \sum_{n=0}^{\ell-1}\frac{\ell C}{N} = \frac{\ell^2 C}{N}$$

Next note that

$$\frac{1}{\ell}\left(\kappa_{N+\ell}-\kappa_N\right)=\frac{N+\ell}{\ell}\sigma_{N+\ell}-\frac{N}{\ell}\sigma_N.$$

Let $\delta_N := \sigma_N - s$. As $\sigma_N \to s$, we must have $\delta_N \to 0$. We can then rewrite

$$\frac{1}{\ell}\left(\kappa_{N+\ell}-\kappa_N\right)=\left(1+\frac{N}{\ell}\right)\delta_{N+\ell}-\frac{N}{\ell}\delta_N+s$$

and hence

$$s_{N-1}-s=\left(1+\frac{N}{\ell}\right)\delta_{N+\ell}-\frac{N}{\ell}\delta_N-\frac{\mathcal{S}_{N,\ell}}{\ell}$$

for all N and all ℓ. We therefore get an upper bound of

$$|s_{N-1}-s|\leq\left(1+\frac{N}{\ell}\right)|\delta_{N+\ell}|+\frac{N}{\ell}|\delta_N|+\frac{\ell C}{N}.$$

Now fix $\varepsilon > 0$. For every $N \geq 1$, we can find $\ell_N > 0$ such that $\ell_N \leq \varepsilon N < \ell_N+1$, and so $\frac{N}{\ell_N+1} \leq \frac{1}{\varepsilon}$. Since our bound holds for all ℓ, it holds for ℓ_N and so we have

$$|s_{N-1}-s|\leq(1+\varepsilon^{-1})|\delta_{N+\ell_N}|+\varepsilon^{-1}|\delta_N|+\varepsilon C$$

As the $\delta_{N+\ell_N}$ and δ_N tend to 0 as $N \to \infty$, we can find make $|s_{N-1} - s|$ arbitrarily small and hence $\lim_{N\to\infty} s_N = s$

4.1 Fejer's counterexample

At this point in the course, all the examples of Fourier series have converged, at least pointwise. We will now construct an example due to Fejer of a continuous function for which the Fejer sums converge, but the partial sums fail to converge. Specifically, we will show that the even 1-periodic function

$$f(x):=\begin{cases}\sum_{k\geq1}\frac{1}{k^2}\sin\left((2^{n^3+1}+1)\pi x\right) & \text{for } 0\leq x\leq\frac{1}{2},\\ -\sum_{k\geq1}\frac{1}{k^2}\sin\left((2^{n^3+1}+1)\pi x\right) & \text{for } -\frac{1}{2}\leq x\leq 0,\\ f(x+1) \text{ for all } x\end{cases}$$

has $\lim_{N\to\infty}(S_{2^{N^3}}f)(0)=\infty$.

We first note that f is a well defined continuous (by the Weierstrauss M-test) even periodic function. As f is even, its Fourier series can be expressed as a cosine series with partial sums

$$(S_Nf)(x)=\sum_{n=0}^{N}A_n\cos(2\pi nx)$$

where

$$A_0=2\int_0^{\frac{1}{2}}f(x)dx\,,\quad A_n=4\int_0^{\frac{1}{2}}f(x)\cos(2\pi nx)dx.$$

Since f is continuous, Fejer's theorem tells us that the Fejer sums converge to $f(x)$, so

$$\lim_{N\to\infty}(\sigma_N f)(0) = f(0) = 0.$$

In order to show that the partial sums do not converge, we first define

$$\lambda_{n,p} := 4\int_0^{\frac{1}{2}} \sin\left((2p+1)\pi x\right)\cos(2\pi n x)dx,$$

$$\Lambda_{N,p} := \sum_{n=1}^{N} \lambda_{n,p}.$$

By swapping the order of summation and integration, we have that

$$A_n = \sum_{k\geq 1} \frac{\lambda_{n,2^{k^3}}}{k^2}$$

and hence the partial sums of f at 0 are given by

$$(S_N f)(0) = A_0 + \sum_{n=1}^{N}\sum_{k\geq 1}\frac{\lambda_{n,2^{k^3}}}{k^2} = A_0 + \sum_{k\geq 1}\frac{\Lambda_{N,2^{k^3}}}{k^2}.$$

It will suffice to show that $\Lambda_{N,p} \geq 0$ and $\Lambda_{N,N} \geq \frac{1}{4\pi}\ln N$. As $\frac{1}{k^2} \geq 0$, we must have that

$$S_{2^{N^3}} f)(0) \geq A_0 + \frac{1}{4\pi N^2}\ln 2^{N^3} = A_0 + \frac{N\ln 2}{4\pi}$$

as a sum of non-negative terms is bounded below by any individual term. This clearly tends to infinity as $N \to \infty$, so $(S_N f)(0)$ cannot converge to a finite limit.

Let us prove our claims about $\Lambda_{N,p}$. First note that

$$\sin\left((2p+1)\pi x\right)\cos(2\pi n x) = \frac{1}{2}\left(\sin((2p+2n+1)\pi x) + \sin((2p-2n+1)\pi x)\right)$$

and hence

$$\lambda_{n,p} = \frac{1}{4\pi}\left(\frac{1}{p+n+\frac{1}{2}} + \frac{1}{p-n+\frac{1}{2}}\right).$$

Then

$$\begin{aligned}
\Lambda_{N,p} &= \frac{1}{4\pi}\left(\sum_{n=0}^{N}\frac{1}{p+n+\frac{1}{2}} + \frac{1}{p-n+\frac{1}{2}}\right) \\
&= \frac{1}{4\pi}\left(\sum_{n=p}^{p+N}\frac{1}{n+\frac{1}{2}} + \sum_{n=p-N}^{p}\frac{1}{n+\frac{1}{2}}\right) \\
&= \frac{1}{4\pi}\left(\frac{1}{p+\frac{1}{2}} + \sum_{n=p-N}^{p+N}\frac{1}{n+\frac{1}{2}}\right).
\end{aligned}$$

If $p > N$, this is a sum of positive terms, and is therefore positive. If $p < N$, then

$$\begin{aligned}
\Lambda_{N,p} &= \frac{1}{4\pi}\left(\frac{1}{p+\frac{1}{2}} + \sum_{n=p-N}^{N-p}\frac{1}{n+\frac{1}{2}} + \sum_{n=N+1-p}^{p+N}\frac{1}{n+\frac{1}{2}}\right) \\
&= \frac{1}{4\pi}\left(\frac{1}{p+\frac{1}{2}} + \frac{1}{N-p+\frac{1}{2}} + \sum_{n=N+1-p}^{p+N}\frac{1}{n+\frac{1}{2}}\right)
\end{aligned}$$

which is the sum of positive terms and is therefore positive.

Finally, if $p = N$, we note that $\frac{1}{x+\frac{1}{2}}$ is monotonically decreasing, and so we can compare the sum $\Lambda_{N,N}$ to the associated integral to get

$$\Lambda_{N,N} \geq \frac{1}{4\pi}\sum_{n=0}^{2N}\frac{1}{n+\frac{1}{2}} \geq \frac{1}{4\pi}\int_0^{2N+1}\frac{dx}{x+\frac{1}{2}},$$

the right hand side of which evaluates to

$$\frac{1}{4\pi}\left(\ln(2N+\frac{3}{2}) - \ln(\frac{1}{2})\right) \geq \frac{1}{4\pi}\ln N.$$

□

5 Solving PDEs using Fourier series

Fourier series methods can be used to find series solutions to a number of families of partial differential equations. We will illustrate five such examples. Our set up will always be as follows. Let

$$u(x,t) : [0,\frac{1}{2}] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$$

be continuously differentiable in each variable. We will assume we have some set of boundary conditions like

$$u(0,t) = u(\frac{1}{2},t) = 0$$

or

$$\partial_t u(0,t) = \partial_t u(\frac{1}{2},t) = 0,$$

and initial conditions given by

$$(\partial_x)^n u(x,0) = f_n(x)$$

for some finite set of $n \geq 0$ and Riemann integrable functions $f_n(x) : [0,\frac{1}{2}] \to \mathbb{R}$. We may, alternatively, have a limiting condition such as

$$\lim_{t\to\infty} u(x,t) = 0$$

or an estimate on the growth of the function $u(x,t)$.

In order to apply Fourier series methods, we need to be able to extend a Riemann integrable function $f : [0, \frac{1}{2}] \to \mathbb{R}$ to a 1-periodic function on $\mathbb{R}$. While this can be done in any number of ways, there are two choices for which the Fourier series will be particularly simple.

5.0.1 Even extension

Define

$$f_{even}(x) := \begin{cases} f(x) & x \in [0, \frac{1}{2}], \\ f(-x) & x \in [-\frac{1}{2}, 0], \\ f_{even}(x+1) = f_{even}(x). \end{cases}$$

If f is continuous, this defines an even 1-periodic continuous function. It has an associated cosine series, and if f is twice continuously differentiable, f_{even} will have absolutely summable Fourier coefficients. This is a good choice if you require boundary conditions like $\partial_t u(0,t) = \partial_t u(\frac{1}{2}, t)$, as $f'(0) = f'(\frac{1}{2}) = 0$ for any f expressed as a cosine series.

Assuming that f_{even} has absolutely summable Fourier coefficients, it has cosine series

$$f(x) = \sum_{n \geq 0} A_n \cos(2\pi n x)$$

where

$$A_0 = 2 \int_0^{\frac{1}{2}} f(x) dx, \qquad A_n = 4 \int_0^{\frac{1}{2}} f(x) \cos(2\pi n x) dx.$$

5.0.2 Odd extension

Define

$$f_{odd}(x) := \begin{cases} f(x) & x \in [0, -\frac{1}{2}], \\ -f(-x) & x \in (-\frac{1}{2}, 0), \\ f_{odd}(x+1) = f_{odd}(x). \end{cases}$$

This only defines a continuous 1-periodic function if $f(0) = f(\frac{1}{2}) = 0$. Otherwise, the extension will be discontinuous and we would need to consider one sided limits when evaluating the Fourier series. The Fourier series can be expressed as a sine series, and is a good choice if you require boundary conditions like $u(0,t) = u(\frac{1}{2}, t)$.

Assuming f_{odd} has absolutely summable Fourier coefficients, it has sine series

$$f(x) = \sum_{n \geq 1} B_n \sin(2\pi n x)$$

where

$$B_n = 4 \int_0^{\frac{1}{2}} f(x) \sin(2\pi n x) dx.$$

5.1 The heat equation with fixed end temperatures

Consider a rod of length $\frac{1}{2}$ such that the ends of the rod are kept at a constant temperature 0. At $t = 0$ the initial temperature distribution is given by a Riemann integrable function $f(x)$ such that $f(0) = f(\frac{1}{2}) = 0$, and such that $\sum_{n\geq 1} |B_n| \leq \infty$. Let the temperature of the rod be given by the function $u(x,t)$. The heat distribution is then determined by the following data, where $\alpha > 0$ is a positive constant related to the conductivity of the rod.

$$\begin{aligned} \alpha\partial_t u = \partial_x^2 u \text{ for all } (x,t) \in [0,\frac{1}{2}] \times (0,\infty), \\ u(0,t) = u(\frac{1}{2},t) = 0 \text{ for all } t > 0 \\ u(x,0) = f(x). \end{aligned}$$

From the above discussion, f is suited to an odd extension, so we take f_{odd} and compute its Fourier expansion as a sine series, which is convergent by assumption. We therefore have

$$u(x,0) = f(x) = \sum_{n\geq 1} B_n \sin(2\pi nx)$$

Based on our computations from the first lecture, a reasonable Ansatz has the t dependence seperated from the x dependence. One such guess would be

$$u(x,t) = \sum_{B_n(t)\sin(2\pi nx)}$$

for some continuously differentiable functions $B_n(t)$ such that

$$\begin{aligned} B_n(0) = B_n, \\ \sum_{n\geq 1} |B_n(t)| \leq \infty. \end{aligned}$$

Such a series is uniformly convergent to a continuous function, and satisfies both our boundary and initial conditions. Filling this guess into our differential equation, and assuming we can swap the order of differentiation and summation (which is valid if our series is uniformly convergent), we get

$$\sum_{n\geq 1} \alpha B_n'(t)\sin(2\pi nx) = \sum_{n\geq 1} -4\pi^2 n^2 B_n(t)\sin(2\pi nx).$$

From the uniqueness of Fourier coefficients, we want that

$$B_n'(t) = \frac{-4\pi^2 n^2}{\alpha} B_n(t)$$

for all $n \geq 1$. This ODE (with initial condition $B_n(0) = B_n$) is solved by

$$B_n(t) = B_n e^{-\frac{4\pi^2 n^2 t}{\alpha}}.$$

As $\sum_{n\geq 1}|B_n(t)| \leq \sum_{n\geq 1}|B_n| \leq \infty$ for all $t \geq 0$, the series

$$u(x,t) = \sum_{n\geq 1} B_n e^{-\frac{4\pi^2 n^2 t}{\alpha}} \sin(2\pi n x)$$

is uniformly convergent and satisfies the conditions of our PDE, giving our solution.

Remark 3. *For differential equations coming from physical systems, a useful sanity check can be to consider the behaviour as $t \to \infty$. In this situation, we see that $u(x,t) \to 0$, which makes physical sense, as we would expect the temperature to average out across the rod, and if the ends are kept at constant temperature 0, energy must be leaving the system.*

5.2 The heat equation with no heat out flux

Consider the same situation as above, but instead of maintaining the ends of the rod at constant temperature, we insulate them so that no heat can enter or exit the system. We take an initial heat distribution $f(x)$ with $f'(0) = f'(\frac{1}{2}) = 0$. The heat distribution is then determined by the following data.

$$\begin{aligned} \alpha\partial_t u &= \partial_x^2 u \text{ for all } (x,t) \in [0, \frac{1}{2}] \times (0,\infty), \\ \partial_x u(0,t) &= \partial_x u(\frac{1}{2}, t) = 0 \text{ for all } t > 0 \\ u(x,0) &= f(x). \end{aligned}$$

Here, f is suited to an even extension, so we compute the Fourier expansion of f_{even} as a cosine series. We assume $\sum_{n\geq 0}|A_n| < \infty$, so that the cosine series is uniformly convergent. As in the previous case, we make an Ansatz of

$$u(x,t) = \sum_{n\geq 0} A_n(t)\cos(2\pi i n x)$$

for some continuously differential functions $A_n(t)$ such that

$$\begin{aligned} A_n(0) &= A_n, \\ \sum_{n\geq 0} |A_n(t)| &\leq \infty. \end{aligned}$$

Such a series is uniformly convergent to a continuous function, and satisfies both our boundary and initial conditions. Filling this guess into our differential equation, and assuming we can swap the order of differentiation and summation (which is valid if our series is uniformly convergent), we get

$$\sum_{n\geq 0} \alpha A_n'(t)\cos(2\pi n x) = \sum_{n\geq 0} -4\pi^2 n^2 A_n(t)\cos(2\pi n x).$$

From the uniqueness of Fourier coefficients, we want that

$$A_n'(t) = \frac{-4\pi^2 n^2}{\alpha} A_n(t)$$

for all $n \geq 1$. This ODE (with initial condition $A_n(0) = A_n$) is solved by

$$A_n(t) = A_n e^{-\frac{4\pi^2 n^2 t}{\alpha}}.$$

As $\sum_{n\geq 0} |A_n(t)| \leq \sum_{n\geq 0} |A_n| \leq \infty$ for all $t \geq 0$, the series

$$u(x,t) = \sum_{n\geq 0} A_n e^{-\frac{4\pi^2 n^2 t}{\alpha}} \cos(2\pi n x)$$

is uniformly convergent and satisfies the conditions of our PDE, giving our solution.

Remark 4. *For differential equations coming from physical systems, a useful sanity check can be to consider the behaviour as $t \to \infty$. In this situation, we see that $u(x,t) \to A_0$, which makes physical sense, as we would expect the temperature to average out across the rod, and no energy enters or exits the system.*

5.3 The inhomogeneous heat equation

Suppose we have a Riemann integrable $f : [0, \frac{1}{2}] \to \mathbb{R}$ such that $f(0) = f(\frac{1}{2}) = 0$, and Riemann integrable $\phi : [0, \frac{1}{2}] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $phi(0,t) = \phi(\frac{1}{2}, t) = 0$. Suppose that the odd extensions have sine series

$$f(x) = \sum_{n\geq 1} B_n \sin(2\pi n x), \quad \phi(x,t) = \sum_{n\geq 1} D_n(t) \sin(2\pi n x)$$

such that $\sum_{n\geq 1} |B_n| \leq \infty$ and $\sum_{n\geq 1} |D_n(t)| \leq \infty$. We think of $\phi(x,t)$ as heat being inputted to the system. The heat distribution is then determined by the data

$$\begin{aligned} \alpha \partial_t u &= \partial_x^2 u + \phi(x,t) \text{ for all } (x,t) \in [0, \frac{1}{2}] \times (0, \infty), \\ u(0,t) &= u(\frac{1}{2}, t) = 0 \text{ for all } t > 0 \\ u(x,0) &= f(x). \end{aligned}$$

From the above discussion, f is suited to an odd extension, so we take f_{odd} and compute its Fourier expansion as a sine series, which is convergent by assumption. We therefore have

$$u(x,0) = f(x) = \sum_{n\geq 1} B_n \sin(2\pi n x)$$

Based on our computations from the first lecture, a reasonable Ansatz has the t dependence seperated from the x dependence. One such guess would be

$$u(x,t) = \sum_{B_n(t)\sin(2\pi nx)}$$

for some continuously differentiable functions $B_n(t)$ such that

$$B_n(0) = B_n,$$
$$\sum_{n\geq 1} |B_n(t)| \leq \infty.$$

Such a series is uniformly convergent to a continuous function, and satisfies both our boundary and initial conditions. Filling this guess into our differential equation, and assuming we can swap the order of differentiation and summation (which is valid if our series is uniformly convergent), we get

$$\sum_{n\geq 1} \alpha B_n'(t)\sin(2\pi nx) = \sum_{n\geq 1} -4\pi^2 n^2 B_n(t)\sin(2\pi nx) + \sum_{n\geq 1} D_n(t)\sin(2\pi nx).$$

Comparing coefficients, we wish to solve

$$B_n'(t) = -\frac{4\pi^2 n^2}{\alpha} B_n(t) + D_n(t)$$

for all n. We therefore need to solve a first order inhomogeneous ODE, and will apply the method of integrating factors (see supplementary note) to determine that

$$B_n(t) = B_n e^{-\frac{4\pi^2 n^2 t}{\alpha}} + \frac{1}{\alpha}\int_0^t e^{-\frac{4\pi^2 n^2 (t-z)}{\alpha}} D_n(z)dz$$

So long as $D_n(z)$ doesn't grow too fast, we will still have $\sum_{n\geq 1} |B_n(t)| \leq \infty$, and so we can conclude that

$$u(x,t) = \sum_{n\geq 1} B_n(t)\sin(2\pi nx)$$

solves our differential equation.

5.4 Solving the wave equation with fixed endpoints

The motion of something like a vibrating string is determined by the wave equations with appropriate boundary and initial conditions. The height of vibrating string with fixed endpoints is determined by the following data:

$$\begin{aligned}
\alpha^2 \partial_t^2 u &= \partial_x^2 u + \phi(x,t) \text{ for all } (x,t) \in [0,\frac{1}{2}] \times (0,\infty),\\
u(0,t) &= u(\frac{1}{2},t) = 0 \text{ for all } t > 0\\
u(x,0) &= f(x),\\
\partial_t u(x,0) &= g(x),
\end{aligned}$$

where α is a non-zero real constant, which we can assume to be positive, and f and g are Riemann integrable functions such with absolutely convergent Fourier expansions

$$f(x) = \sum_{n\geq 1} B_n \sin(2\pi n x),$$
$$g(x) = \sum_{n\geq 1} C_n \sin(2\pi n x).$$

Given the boundary conditions, a reasonable Ansatz would be

$$u(x,t) = \sum_{n\geq 1} U_n(t) \sin(2\pi n x)$$

where

$$U_n(0) = B_n,$$
$$U_n'(0) = C_n,$$
$$\sum_{n\geq 1} |U_n(t)| \leq \infty.$$

Such a series is uniformly convergent to a continuous function, and satisfies both our boundary and initial conditions. Filling this guess into our differential equation, and assuming we can swap the order of differentiation and summation (which is valid if our series is uniformly convergent), we get

$$\sum_{n\geq 1} \alpha^2 U_n''(t) \sin(2\pi n x) = -4\pi^2 n^2 U_n(t) \sin(2\pi n x).$$

Comparing coefficients, we want U_n to solve the ODE

$$U_n''(t) = -\left(\frac{2\pi n}{\alpha}\right) U_n(t).$$

Given our initial conditions, this has solution

$$U_n(t) = B_n \cos(\frac{2\pi n}{\alpha} x) + \frac{\alpha}{2\pi n} C_n \sin(\frac{2\pi n}{\alpha} x).$$

The triangle inequality tells us that

$$\sum_{n\geq 1} |U_n(t)| \leq \sum_{n\geq 1} |B_n| + \sum_{n\geq 1} |C_n| \leq \infty$$

and so we conclude that

$$u(x,t) = \sum_{n\geq 1} \left(B_n \cos(\frac{2\pi n}{\alpha} x) + \frac{\alpha}{2\pi n} C_n \sin(\frac{2\pi n}{\alpha} x) \right) \sin(2\pi n x)$$

is a solution to the wave equation.

Remark 5. *While it might seem initially unphysical that $u(x,t)$ is periodic in time when a plucked string gradually stops vibrating, in an ideal world there would be no energy lost from the system, so this repeating behaviour makes sense here.*

5.5 A variation on the Laplace equation

The Laplace equation usually refers to a differential equation

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0$$

for a function

$$u : \{(x,y) \mid x^2 + y^2 \leq 1\} \to \mathbb{R}$$

on a circle. This differential equation has connections to electrostatics, gravitation, fluid dynamics, and complex analysis. We will consider a modified version that could be used to model a vibrating string with energy loss. Specifically, given $\alpha > 0$ and f as for the wave equation, we will find $u : [0, \frac{1}{2}] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\begin{aligned}
-\alpha^2 \partial_t^2 u &= \partial_x^2 u + \phi(x,t) \text{ for all } (x,t) \in [0, \frac{1}{2}] \times (0, \infty), \\
u(0,t) &= u(\frac{1}{2}, t) = 0 \text{ for all } t > 0 \\
u(x,0) &= f(x), \\
\lim_{t \to \infty} u(x,t) &= 0.
\end{aligned}$$

Proceeding as before, we find that a candidate for $u(x,t)$ is given by the series

$$u(x,t) = \sum_{n \geq 1} \left(P_n e^{\frac{2\pi n t}{\alpha}} + Q_n e^{-\frac{2\pi n t}{\alpha}} \right) \sin(2\pi n x)$$

From the condition that $u(x,t) \to 0$, we see that we should $P_n = 0$ for every n, as otherwise this exponential term would dominate the growth as $t \to \infty$. This also ensures absolute summability of the Fourier coefficients and hence uniform convergence. Therefore

$$u(x,t) = \sum_{n \geq 1} B_n e^{-\frac{2\pi n t}{\alpha}} \sin(2\pi n x)$$

is the desired solution. This is an example of a system exhibiting exponential decay - in a number of practical situations, you may not have the necessary number of initial conditions to uniquely determine a solution, but imposing growth constraints such as exponential decay can also help us in determining a solution.

Week 3

6 Inner product spaces

Definition 5. *Let V be a complex vector space, possibly infinite dimensional. An inner product on V is a function*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

such that

1. *Linearity:*
$$\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle,$$

2. *Conjugate linearity:*
$$\langle v, \lambda_1 w_1 + \lambda_2 w_2 \rangle = \bar{\lambda}_1 \langle v, w_1 \rangle + \bar{\lambda}_2 \langle v, w_2 \rangle,$$

3. *Conjugate symmetric:*
$$\langle v, w \rangle = \overline{\langle w, v \rangle},$$

4. *Positive semidefinite:*
$$\|f\|^2 := \langle f, f, \rangle \in \mathbb{R}_{\geq 0},$$

5. *Non-degenerate: $\|f\| = 0$ if and only if $f = 0$.*

A complex vector space V with a given inner product is called an inner product space. The function
$$\| \cdot \| : V \to \mathbb{R}_{\geq 0}$$
is called a norm.

Example 3. *Let $V = \mathbb{C}^n$. The complex dot product*

$$\langle v, w \rangle := \sum_{k=1}^{n} v_k \bar{w}_k$$

defines an inner product on V.

Example 4. *Define the space of square summable sequences to be*

$$\ell^2 := \{(a_n)_{n \geq 1} \mid a_n \in \mathbb{C}, \sum_{n \geq 1} |a_n|^2 < \infty\}.$$

This is a vector space equipped with component-wise addition and scalar multiplication.

Proof. Note that

$$\lambda(a_n)_{n\geq 1} = (\lambda a_n)_{n\geq 1}, \quad \sum_{n\geq 1} |\lambda a_n|^2 = |\lambda|^2 \sum_{n\geq 1} |a_n|^2 < \infty.$$

Hence ℓ^2 is closed under scalar multiplication. To see that it is closed under addition, it suffices to show that

$$\sum_{n\geq 1} |a_n + b_n|^2 = \sum_{n\geq 1} |a_n|^2 + 2|a_n b_b| + |b_n|^2 < \infty$$

Note that, since absolute values are real,

$$|a_n|^2 - 2|a_n b_n| + |b_n|^2 = (|a_n| - |b_n|)^2 \geq 0$$

and hence $2|a_n b_n| \leq |a_n|^2 + |b_n|2$. Thus

$$\sum_{n\geq 1} |a_n + b_n|^2 \leq \sum_{n\geq 1} 2|a_n|^2 + 2|b_n|^2 = 2\sum_{n\geq 1} |a_n|^2 + 2\sum_{n\geq 1} |b_n|^2 < \infty.$$

□

We define an inner product on ℓ^2 by

$$\langle (a_n)_{n\geq 1}, (b_n)_{n\geq 1} \rangle := \sum_{n\geq 1} a_n \bar{b}_n.$$

This converges because it converges absolutely, from the same argument used in showing ℓ^2 is a vector space, and it is easy to check that it satisfies the axioms of an inner product.

Remark 6. *In the exercise session, we will establish the convergence of the ℓ^2 inner product by expressing it in terms of norms. This is a useful approach when given a norm on a vector space and a candidate for an inner product where the convergence of the inner product is unknown.*

Example 5. *Let*

$$\mathcal{C}_1^0 := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ continuous and } 1\text{-periodic}\}.$$

This is easily seen to be a vector space with addition and scalar multiplication defined pointwise:

$$(f+g)(x) := f(x) + g(x),$$
$$(\lambda f)(x) := \lambda f(x).$$

We define a map $\mathcal{C}_1^0 \times \mathcal{C}_1^0 \to \mathbb{C}$ by

$$\langle f, g \rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)} dx.$$

It is easy to check that this satisfies the first four properties of an inner product. To see that this defines a non-degenerate inner product, consider non-zero $f \in \mathcal{C}_1^0$*. As* f *is non-zero, there is a point* $x_0 \in \mathbb{R}$ *such that* $f(x_0) \neq 0$*. Without loss of generality, we may assume* $x_0 = 0$*. By continuity, there exists* $\delta > 0$ *such that*

$$|f(x)|^2 \geq \frac{|f(0)|^2}{2} > 0$$

for all $|x| < \delta$*. As* $|f(x)|^2 \geq 0$ *for all* x*, we must have*

$$\|f\|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx \geq \int_{|x|<\delta} |f(x)|^2 dx \geq \int_{|x|<\delta} \frac{|f(0)|^2}{2} dx > 0.$$

Hence $\|f\| = 0$ *if and only if* $f = 0$*.*

Remark 7. *We could alternatively consider*

$$V := \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ Riemann integrable and } 1\text{-periodic}\},$$

equipped with the same map. Unfortunately

$$\langle f, g \rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)} dx$$

is not non-degenerate on V*: there exist integrable* 1*-periodic non-zero functions with square integral* 0*. For example*

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Z}, \\ 1 & x \in \mathbb{Z}. \end{cases}$$

However, many results about the space of continuous 1*-periodic functions can be lifted to results about integrable* 1*-periodic functions via continuous approximation.*

A short, but often useful observation is that for any v, w in an inner product space V,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2 + 2\Re\langle v, w \rangle.$$

In the case of $V = \mathbb{C}^n$, and v, w real vectors, this is precisely the statement of the generalised Pythagorean theorem

$$a^2 + b^2 = c^2 - 2ab\cos(\theta).$$

It is often easier to compute norms than it is to compute inner products. If we only need to obtain a bound on an inner product, a valuable tool is the Cauchy Schwartz inequality.

Lemma 4 (Cauchy Schwartz inequality)**.** *Let* V *be an inner product space. Then for all* $v, w \in V$*,*

$$|\langle v, w \rangle| \leq \|v\|\|w\|$$

with equality if and only if there exists $\alpha \in \mathbb{C}$ *such that* $v = \alpha w$*.*

Proof. The claim is clearly true if $\|w\| = 0$ or $|\langle v, w\rangle| = 0$, we we will assume otherwise. Let $\eta = \frac{|\langle v,w\rangle|}{\langle v,w\rangle}$ and note that

$$0 \le \|t\eta v - w\|^2 \text{ for all } t \in \mathbb{R}.$$

Expanding this out, we see that

$$\begin{aligned} 0 &\le t^2\|v\|^2 - 2t\Re\eta\langle v, w\rangle + \|w\|^2 \\ &= t^2\|v\|^2 - 2t\Re|langlev, w\rangle| + \|w\|^2 \\ &= t^2\|v\|^2 - 2t|langlev, w\rangle| + \|w\|^2 \\ &= \left(t\|v\| - \frac{|\langle v, w\rangle|}{\|v\|}\right)^2 + \|w\|^2 - \frac{|\langle v, w\rangle|^2}{\|v\|^2} \end{aligned}$$

for all real t. Taking $t = \frac{|\langle v,w\rangle|}{\|v\|}$ to get that

$$0 \le \|w\|^2 - \frac{|\langle v, w\rangle|^2}{\|v\|^2}$$

or

$$|\langle v, w\rangle|^2 \le \|v\|^2\|w\|^2.$$

Since all quantities are positive, we can take the square root to obtain the Cauchy-Schwartz inequality. We obtain equality if and only if

$$0 = \|t\eta v - w\|$$

i.e there exists $\alpha \in \mathbb{C}$ such that $v = \alpha w$. □

A corollary of this is that the triangle inequality holds in all inner product spaces

Corollary 6. *For any v, w in an inner product space V $\|v + w\| \le \|v\| + \|w\|$.*

Proof. Note that

$$\|v + w\|^2 = \|v\|^2 + 2\Re\langle v, w\rangle + \|w\|^2$$

and that $\Re\langle v, w\rangle \le |\langle v, w\rangle|$. By the Cauchy-Schwartz inequality

$$|\langle v, w\rangle| \le \|v\|\|w\|.$$

Hence

$$\|v + w\|^2 \le \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

The triangle inequality is the square root of this inequality. □

Example 6. *For $V = \mathbb{C}^2$ and $v, w \in \mathbb{R}^2$, the Cauchy-Schwartz inequality tells us that*

$$|\langle v, w\rangle| \le \|v\|\|w\|$$

For a pair of vectors in $\mathbb{R}^2$, the inner product is the dot product is $|\langle v, w\rangle| = \|v\|\|w\|\cos(\theta)$ where θ is the smaller angle between v and w. Hence $|\cos(\theta)| \le 1$.

Example 7. *Let V be the space of continuous functions on $[0,1]$ equipped with the inner product*

$$\langle f,g\rangle := \int_0^1 f(x)\overline{g(x)}dx.$$

This lets us bound difficult integrals via the Cauchy-Schwartz inequality:

$$0.592\ldots = \left|\int_0^1 \sqrt{x\cos x}dx\right| \le \left(\int_0^1 xdx\right)^{\frac{1}{2}} \left(\int_0^1 \cos xdx\right)^{\frac{1}{2}} = \frac{\sin(1)}{2} = 0.648\ldots$$

6.1 Orthonormal sets and best approximation

Definition 6. *Let V be an inner product space. A pair of vectors (v,w) are called orthogonal $\langle v,w\rangle = 0$. A set of vectors S is called orthogonal each pair of distinct vectors $v \neq w \in S$ are othogonal. A set of vectors S is called orthonormal if it is orthogonal and $\|v\| = 2$ for every $v \in S$.*

Lemma 5 (Pythagorean theorem)**.** *If v,w are orthogonal vectors in an inner product space V, then*

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2.$$

Proof. As noted previously

$$\|v+w\|^2 = \|v\|^2 + 2\Re\langle v,w\rangle + \|w\|^2.$$

If $\langle v,w\rangle = 0$, we obtain the desired result. □

Example 8. *The standard basis of $\mathbb{C}^n$ is an orthonormal set.*

Example 9. *Let $\mathcal{C}_1^0$ be the space. The set $\{e_n(x) = e^{2\pi inx}\}$ is an orthonormal set.*

Example 10. *Let V be the inner product space of continuous functions on $[-1,1]$ with inner product*

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

Then $S = \{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal set.

Given an orthonormal set in an inner product space V, one might hope that we can execute a Gram-Schmidt-like producedure to express any vector $v \in V$ in terms of the orthonormal set. This is possible in a finite dimensional vector space, but it is not always possible in an infinite dimensional vector space. However, we can show that such a procedure constructs a good approximation.

Proposition 3 (Best approximation)**.** *Let V be an inner product space and let $\{e_n\}_{n\in S}$ be a finite set of orthonormal vectors in V and choose $v \in V$. Define $c_n := \langle v, e_n\rangle$. Then for any collection of complex numbers $\{b_n\}_{n\in S}$*

$$\|v - \sum_{n\in S} c_n e_n\| \le \|v - \sum_{n\in S} b_n e_n\|$$

with equality if and only if $b_n = c_n$ *for every* $n \in S$. *That is to say, the best approximation of* v *is given by* $\sum_{n\in S} c_n e_n$.

Proof. Note that, for any $\{a_n\}_{n\in S}$,

$$\begin{aligned}\langle v - \sum_{n\in S} c_n e_n, \sum_{n\in S} a_n e_n\rangle &= \sum_{n\in S} \bar{a}_n \langle v, e_n\rangle - \sum_{m,n\in S} c_n \bar{a}_m \langle e_n, e_m\rangle \\ &= \sum_{n\in S} c_n \bar{a}_n + \sum_{n\in S} c_n \bar{a}_n = 0.\end{aligned}$$

Hence

$$\begin{aligned}\|v - \sum_{n\in S} c_n e_n + \sum_{n\in S} a_n e_n\|^2 &= \|v - \sum_{n\in S} c_n e_n\|^2 + \sum_{n\in S} \|a_n e_n\|^2 \\ &= \|v - \sum_{n\in S} c_n e_n\|^2 + \sum_{n\in S} |a_n|^2\end{aligned}$$

and thus

$$\|v - \sum_{n\in S} c_n e_n + \sum_{n\in S} a_n e_n\|^2 \geq \|v - \sum_{n\in S} c_n e_n\|^2.$$

Let $a_n = c_n - b_n$ to get

$$\|v - \sum_{n\in S} b_n e_n\|^2 \geq \|v - \sum_{n\in S} c_n e_n\|^2.$$

□

We may similarly use the Pythagorean theorem to prove Bessel's inequality.

Proposition 4 (Bessel's inequality)**.** *Let* V *be an inner product space and let* $\{e_n\}_{n\in S}$ *be a finite set of orthonormal vectors in* V. *Let* $v \in V$. *Define* $c_n := \langle v, e_n\rangle$. *Then*

$$\sum_{n\in S} |c_n|^2 \leq \|v\|^2.$$

Proof. Write $v = v - \sum_{n\in S} c_n e_n + \sum_{n\in S} c_n e_n$. Applying the Pythagorean theorem and using $\|e_n\| = 1$, we get

$$\|v\|^2 = \|v - \sum_{n\in S} c_n e_n\|^2 + \sum_{n\in S} |c_n|^2 \|e_n\|^2$$

and hence

$$\|v\|^2 \geq \sum_{n\in S} |c_n|^2.$$

□

While the proofs of both of the above assume S is finite, there is nothing stopping us from taking a limit over growing S to obtain that both the best approximation theorem and Bessel's inequality hold for infinite ordered sets i.e. $S = \mathbb{Z}$.

7 Applications to Fourier analysis

From here, we will consider only the case of $V = \mathcal{C}_1^0$, continuous 1-periodic functions equipped with inner product

$$\langle f, g \rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)}dx.$$

We will refer to

$$\|f\| = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx$$

as the L^2 norm.

We have three notable infinite orthonormal sets in $\mathcal{C}_1^0$.

7.0.1 Exponential functions

Consider the orthonormal set $\{e_n(x) := e^{2\pi i n x}\}_{n\in\mathbb{Z}}$. Then

$$\langle f, e_n \rangle = c_f(n).$$

Taking limits of Bessel's inequality says that

$$\sum_{n\in\mathbb{Z}} |c_f(n)|^2 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx.$$

The best approximation theorem says that

$$(S_N f)(x) := \sum_{n=-N}^{N} c_f(n) e^{2\pi i n x}$$

is the best approximation of f in trigonometric polynomials of degree at most N with respect to the L^2 norm.

7.0.2 Sine series

Consider the orthonormal set $\{e_n(x) := 2\sin(2\pi n x)\}_{n\geq 1}$. Then

$$\langle f, e_n \rangle = B_n.$$

Bessel's inequality says that

$$\sum_{n\geq 1} |B_n|^2 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx.$$

The best approximation theorem says that

$$\sum_{n\geq 1} B_n \sin(2\pi n x)$$

is the best approximation of f via sine series of length at most N with respect to the L^2 norm.

7.0.3 Cosine series

Consider the orthonormal set $\{e_n(x) := 2\cos(2\pi nx)\}_{n\geq 0}$. Then

$$\langle f, e_n \rangle = A_n.$$

Bessel's inequality says that

$$\sum_{n\geq 0} |A_n|^2 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx.$$

The best approximation theorem says that

$$\sum_{n\geq 0} A_n \cos(2\pi nx)$$

is the best approximation of f via cosine series of length at most N with respect to the L^2 norm.

Remark 8. *Note that at no point in the proof of Bessel's inequality or the best approximation theorem did we use non-degeneracy of the inner product. Therefore both these results hold for the space $\mathcal{R}^1$ of Riemann integrable* 1-*periodic functions, and so do the above discussions.*

The best approximation theorem suggests that we should expect $(S_N f) \to f$ as $N \to \infty$ in the L^2 norm, i.e.

$$\|f - S_N f\| \to 0$$

for continuous 1-periodic f. This is precisely the case, and tells us that all the information regarding integrals of continuous 1-periodic functions are encoded in the Fourier series.

Proposition 5. *For f a continuous* 1-*periodic function* $\|f - S_N f\| \to 0$.

Proof. Given $\varepsilon > 0$, we showed in Week 1, Corollary 4, we can find a trigonometric polynomial of degree N

$$p(x) = \sum_{n=-N}^{N} p_n e^{2\pi inx}$$

such that

$$|f(x) - p(x)| \leq \varepsilon$$

for all $x \in [-\frac{1}{2}, \frac{1}{2}]$. Squaring this and integrating gives that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x) - p(x)|^2 dx \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \varepsilon^2 dx = \varepsilon^2$$

which is to say

$$\|f - p\|^2 \leq \varepsilon^2.$$

Taking the square root and applying the best approximation theorem, we see that

$$\|f - S_M f\| \leq \|f - p\| \leq \varepsilon$$

whenever $M > N$. The claim then follows, as ε was arbitrary. □

Fact 1. *For f any Riemann integrable 1-periodic function, $\lim_{N\to\infty} \|f - S_N f\| = 0$. This follows as we can approximate f arbitrarily well by continuous functions, though we will not prove this.*

Exercise 5. *Assuming the above fact, show that*

$$\langle f, g\rangle = \lim_{N\to\infty} \langle S_N f, g\rangle$$

for any Riemann integrable 1-periodic f and g. As a hint, consider writing the inner product in terms of norms.

Hence deduce Parseval's identity for Riemann integrable f:

$$\sum_{n\in\mathbb{Z}} |c_f(n)|^2 = \|f\|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx.$$

We will prove Parseval's identity for Riemann integrable f without assuming this fact. Recall the definition of the circular convolution of two Riemann inegrable 1 periodic functions f, g is defined by the integral

$$(f * g)(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(y)g(x - y)dy.$$

As we showed in exercise session 3, convolution is a commutative, associative product on the space of Riemann integrable functions, and satisfies

$$c_{(f*g)}(n) = c_f(n)c_g(n).$$

Furthermore, $(f * g)(x)$ is a continuous 1-periodic function.

Theorem 3. *For any Riemann integrable 1-periodic $f : \mathbb{R} \to \mathbb{C}$*

$$\sum_{n\in\mathbb{Z}} |c_f(n)|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx$$

Proof. By Bessel's inequality

$$\sum_{n\in\mathbb{Z}} |c_f(n)|^2 = \lim_{N\to\infty} \sum_{n=-N}^{N} |c_f(n)|^2 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 dx.$$

Define $\tilde{f}(x) := \overline{f(-x)}$ and note that

$$\begin{aligned} c_{\tilde{f}}(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{f(-x)} e^{-2\pi i n x} dx \\ &= \overline{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(-x) e^{2\pi i n x} dx} \\ &= \overline{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} dx} = \overline{c_f(n)} \end{aligned}$$

and hence

$$|c_f(n)|^2 = c_f(n)\overline{c_f(n)} = c_f(n)c_{\tilde{f}}(n) = c_{(f*\tilde{f})}(n).$$

Thus, Bessel's inequality implies the convergence of $\sum_{n\in\mathbb{Z}} |c_{(f*\tilde{f})}(n)|$. Thus the Fourier series converges to $(f * \tilde{f})(x)$. In particular

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |c_f(n)|^2 &= \sum_{n\in\mathbb{Z}} c_{(f*\tilde{f})}(n) \\ &= (f*\tilde{f})(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(y)\tilde{f}(-y)dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(y)\overline{f(y)}dy = \|f\|. \end{aligned}$$

□

Remark 9. *We only require that $(f * g)(x)$ be continuous at 0 for the above argument to work.*

Corollary 7. *For integrable 1-periodic f, g*

$$\sum_{n\in\mathbb{Z}} c_f(n)\overline{c_g(n)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)}dx.$$

Proof. We can either apply a similar convolution argument to Parseval's identity, or apply Parseval's identity and the equality

$$\langle f, g\rangle = \frac{1}{4}\left((\|f+g\|^2 - \|f-g\|^2) + i(\|f+ig\|^2 - \|f-ig\|^2)\right).$$

□

Exercise 6. *Prove Corollary 7 by direct computation using Fact 1.*

We conclude with a final corollary of Bessel's inequality.

Lemma 6 (Riemann-Lebesugue Lemma)**.** *If f is a Riemann integrable 1-periodic function, then $c_f(n) \to 0$ as $|n| \to \infty$.*

Proof. Bessel's inequality implies that

$$\sum_{n\in\mathbb{Z}} |c_f(n)|^2 \leq \|f\| < \infty.$$

Doubly infinite series converge if and only if its terms tend to 0 as $|n| \to \infty$. Therefore $|c_f(n)|^2 \to 0$ and so $c_f(n) \to 0$. $\square$

Week 4

8 Poisson summation and the Fourier transform

Previously, we have shown that an enormous amount of information about periodic functions is encoded in their Fourier series. These same methods do not apply immediately to non-periodic functions. However, we can still capture some information about such functions via Fourier series computations.

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function and that there exists $C > 0$ such that $f(x)$ is absolutely integrable

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty,$$

for example if

$$|f(x)| \leq \frac{C}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

We can construct a 1-periodic function associated to f as follows. Define

$$\phi_f(x) := \sum_{n \in \mathbb{Z}} f(x+n).$$

As

$$\phi_f(x+1) = \sum_{n \in \mathbb{Z}} f(x+n+1) = \sum_{n \in \mathbb{Z}} f(x+n) = \phi_f(x),$$

the function ϕ_f is 1-periodic if it is well defined. As such, it suffices to show that it is well defined on $[-\frac{1}{2}, \frac{1}{2}]$. But in the case where

$$|f(x)| \leq \frac{C}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

we have that

$$|f(x+n)| \leq \frac{C}{1+n^2}$$

for all $x \in [-\frac{1}{2}, \frac{1}{2}]$, and so by the Weierstrauss M-test, the series is uniformly convergent. Thus, ϕ_f is well defined and continuous. More generally, we can establish convergence by an integral comparison test.

The Fourier coefficients of ϕ_f are given by

$$\begin{aligned}
c_{\phi_f}(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_f(x) e^{-2\pi i n x} dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} f(x+n) e^{-2\pi i n x} dx \\
&= \sum_{n \in \mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+n) e^{-2\pi i n x} dx \\
&= \sum_{n \in \mathbb{Z}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x) e^{-2\pi i n x} dx \\
&= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx.
\end{aligned}$$

Note that, since this is an infinite integral, there are no periodicity constraints on $e^{-2\pi i n x}$, and so we can extend the definition of Fourier coefficients of ϕ_f to a function on the real line.

Definition 7. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous absolutely integrable function. The Fourier transform of f is defined as*

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx.$$

Our prior calculations then just say that $\hat{f}(n) = c_{\phi_f}(n)$. Now suppose $\hat{f}(\xi)$ is also absolutely integrable. Suppose further that

$$\sum_{n \in \mathbb{Z}} |c_{\phi_f}(n)| < \infty$$

and so the Fourier series converges to $\phi_f(x)$ uniformly: Thus we have mostly established the following.

Theorem 4 (Poisson Summation Formula)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, absolutely integrable function with absolutely integrable Fourier transform $\hat{f}$. Then, if $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$,*

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

In particular, at $x = 0$

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m).$$

Proof. As noted above, $\hat{f}(m) = c_{\phi_f}(m)$ are the Fourier coefficients of the continuous 1-periodic function ϕ_f. If $\hat{f}$ is absolutely integrable, then

$$\sum_{n \in \mathbb{Z}} |c_{\phi_f}(n)| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$$

and so

$$\phi_f(x) = \sum_{n\in\mathbb{Z}} c_{\phi_f}(n)e^{2\pi inx} = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$$

for all x. Thus

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}.$$

□

Thus, the Fourier transform of f at integer points captures some sort of discrete information about f. One might hope that by considering $\hat{f}(\xi)$ for all $\xi \in \mathbb{R}$, one might capture more information about f at all real points. In order to make this precise, we will need a bit more theory, so we will first give some examples of Fourier transforms.

Example 11. *Let $f(x) = e^{-|x|}$. This is absolutely integrable, and so has a well defined Fourier transform.*

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-|x|}e^{-2\pi i\xi x}dx \\ &= \int_{0}^{\infty} e^{-(1+2\pi i\xi)x}dx + \int_{-\infty}^{0} e^{(1-2\pi i\xi)x}dx \\ &= \frac{1}{1+2\pi i\xi} + \frac{1}{1-2\pi i\xi} = \frac{2}{1+4\pi^2\xi^2}.\end{aligned}$$

This is of moderate growth and therefore absolutely integrable. Thus we can apply Poisson summation to get that

$$2\sum_{n\in\mathbb{Z}} \frac{1}{1+4\pi^2n^2} = \sum_{n\in\mathbb{Z}} e^{-|n|}.$$

Rearranging this we find that

$$2 + 4\sum_{n=1}^{\infty} \frac{1}{1+4\pi^2n^2} = 1 + \frac{2e^{-1}}{1-e^{-1}} = coth(\frac{1}{2}).$$

Not all Fourier transforms are this easy to compute though. A particularly important, but tricky, example is the Fourier transform of the Gaussian.

Example 12. *Let $f(x) = e^{-\pi x^2}$. This is referred to as the normalised Gaussian function. From the decay of the exponential, this is clearly absolutely integrable, though there is a trick to computing the integral. We begin by considering the* *square of the integral.*

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi x^2 - \pi y^2} dxdy \\ &= \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)} dxdy \\ &= \int_{\mathbb{R}^2} re^{-\pi r^2} drd\theta \\ &= 2\pi \int_0^{\infty} re^{-\pi r^2} dr = 1.\end{aligned}$$

To compute the Fourier transform, we also need a trick. We write

$$\begin{aligned}\hat{f}(\xi) &\int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i \xi x} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2 - \pi\xi^2} dx \\ &- e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2} dx.\end{aligned}$$

Let $\phi(\xi) = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2} dx$ *and note that*

$$\begin{aligned}\phi(0) &= \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1, \\ \phi'(\xi) &= \int_{-\infty}^{\infty} -2\pi i(x+i\xi)e^{-\pi(x+i\xi)^2} dx.\end{aligned}$$

We recognise the integrand in $\phi'(\xi)$ *is also a derivative and therefore*

$$\begin{aligned}\phi'(\xi) &= \int_{-\infty}^{\infty} -2\pi i(x+i\xi)e^{-\pi(x+i\xi)^2} dx \\ &= i\int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-pi(x+i\xi)^2}\right) dx\end{aligned}$$

$$= i\left[e^{-\pi(x+i\xi)^2}\right]_{-\infty}^{\infty} = 0$$

for all $\xi \in \mathbb{R}$. *Thus* $\phi(\xi) = 1$ *for all* $\xi \in \mathbb{R}$ *and so*

$$\hat{f}(\xi) = e^{-\pi\xi^2}.$$

The normalised Gaussian is self dual!

Example 13. *The Poisson summation formula for the normalised Gaussian is extremely important in analytic number theory. We define the theta function*

$$\theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi t n^2} = \sum_{n\in\mathbb{Z}} g_t(n)$$

where $g_t(x) := e^{-\pi t x^2}$. From our above calculations and a change of variables, we know that

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{t}} e^{-\pi \frac{\xi^2}{t}} = \frac{1}{\sqrt{t}} g_{1/t}(\xi)$$

and so the Poisson summation formula gives that

$$\theta(t) = \sum_{n\in\mathbb{Z}} g_t(n) = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} g_{1/t}(n) = \frac{\theta(\frac{1}{t})}{\sqrt{t}}.$$

This inversion property for the theta function is then used to prove the functional equation for the Riemann zeta function

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$$

which allows us to extend the zeta function to almost the entire complex plane:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

In particular, this can be used to give meaning to the divergent sum

$$\frac{-1}{12} = \zeta(-1) \text{``} = \sum n \geq 1n''.$$

9 The Fourier inversion formula

In the case of Fourier series, if f was a continuous 1-periodic function whose Fourier series was absolutely summable, we could recover f from $(c_f(n))_{n\in\mathbb{Z}}$. Thus we obtained a bijection between a set of "nice enough" functions and "nice enough" sequences of complex numbers. We will show that a continuous analogue of the same construction gives us that the Fourier transform is a bijection for "nice enough" functions.

Definition 8. *Suppose $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous and absolutely integrable. The inverse Fourier transform of $\hat{f}$ is defined as*

$$f^{\vee}(x) := \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \hat{\hat{f}}(-x).$$

Theorem 5. *Suppose f is a continuous, bounded, and absolutely integrable function with continuous, absolutely integrable Fourier transform $\hat{f}$. Then $f = f^{\vee}$, i.e.*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

for all x.

Remark 10. *While we will only prove the above theorem in the above case, note that this includes functions of moderate growth, i.e. those bounded by something like* $\frac{C}{1+x^2}$.

We will prove this using a continuous version of our Key Lemma about Dirac families. Let us first define a continuous Dirac family.

Definition 9. *A (continuous) Dirac family is a collection of non-negative even functions on the real line*

$$\{\rho_T : \mathbb{R} \to \mathbb{R} \mid T \in (0, L)\},$$

where L *is possibly infinite, satisfying*

$$\int_{-\infty}^{\infty} \rho_T(y)dy = 1$$

for all T *and for every* $\delta > 0$

$$\lim_{T \to L} \int_{|y| \geq \delta} \rho_T(y)dy = 0.$$

Lemma 7 (Continuous Key Lemma)**.** *If* $f : \mathbb{R} \to \mathbb{C}$ *is continuous, bounded and absolutely integrable, and* $\{\rho_T\}_{T \in (0,L)}$ *is a Dirac family, then*

$$\lim_{T \to L} \int_{-\infty}^{\infty} \rho_T(y) f(x-y)dy = f(x).$$

If, furthermore, f *is uniformly continuous, then this limit converges uniformly.*

Proof. We fix some x. Then, for all $\varepsilon > 0$ there exists $\delta_x > 0$ such that

$$|f(x-y) - f(x)| < \varepsilon$$

for all $|y| < \delta_x$. Then

$$\begin{aligned} |\int_{-\infty}^{\infty} \rho_T(y) f(x-y)dy - f(x)| &= |\int_{-\infty}^{\infty} \rho_T(y) \left(f(x-y) - f(x)\right) dy| \\ &\leq \int_{|y|<\delta_x} \rho_T(y)|f(x-y) - f(x)|dy \\ &\quad + \int_{|y|\geq\delta_x} \rho_T(y) \left(|f(x-y)| + |f(y)|\right) dy \end{aligned}$$

In the first integral

$$|f(x-y) - f(x)| < \varepsilon$$

and thus the first integral is bounded by ε, as the integral of $\rho_T(y)$ across any interval is at most 1. To bound the second interval, since f is bounded, we must have that

$$\int_{|y|\geq\delta_x} \rho_T(y) \left(|f(x-y)| + |f(y)|\right) dy \leq 2\|f\|_\infty \int_{|y|\geq\delta_x} \rho_T(y)dy$$

which tends to 0 as $T \to L$. Thus, for T close enough to L, it is bounded by ε. Thus

$$|\int_{|y|\geq\delta_x} \rho_T(y)f(x-y)dy - f(x)| \leq 2\varepsilon$$

for T close enough to L and hence

$$\lim_{T\to L}\int_{|y|\geq\delta_x} \rho_T(y)f(x-y)dy = f(x).$$

Note that "T close enough to L" depends on x via δ_x. However if f is uniformly continuous, then we may take $\delta_x = \delta$ for some $\delta > 0$ and all x, and thus

$$\lim_{T\to L}\sup_{x\in\mathbb{R}}|\int_{|y|\geq\delta_x} \rho_T(y)f(x-y)dy - f(x)| = 0$$

and we get that this limit converges uniformly. $\square$

We require one final lemma before proving the Inversion theorem. Specifically, we need a Dirac family.

Lemma 8. *The family* $\{\rho_T(x) = Te^{-\pi T^2x^2}\}_{T\in(0,\infty)}$ *is a Dirac family.*

Proof. Clearly, $\rho_T(x)$ is even an non-negative. By a change of variables

$$\int_{-\infty}^{\infty}\rho_T(y)dy = T\int_{-\infty}^{\infty}e^{-\pi T^2y^2}dy = \int_{-\infty}^{\infty}e^{-\pi u^2}du = 1.$$

By the same change of variables

$$\int_{|y|\geq\delta}Te^{-\pi T^2y^2}dy = \int_{|u|\geq T\delta}e^{-\pi u^2}du$$

which tends to 0 as $T\to\infty$. $\square$

Proof of Theorem 5. The assumptions of our Continuous Key Lemma apply here, so we must have that

$$\begin{aligned} f(x) &= \lim_{T\to\infty}\int_{-\infty}^{\infty}Te^{-\pi T^2y^2}f(x-y)dy \\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty}Te^{-\pi T^2(z-x)^2}f(z)dz. \end{aligned}$$

Noting that

$$e^{-\pi T^2(z-x)^2} = \int_{-\infty}^{\infty}e^{-\pi q^2}e^{-2\pi iq(T(z-x))}dq$$

we see that

$$\begin{aligned}
f(x) &= \lim_{T\to\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Te^{-\pi q^2} e^{2\pi iqTx - 2\pi iqTz} f(z)dqdz \\
&= \lim_{T\to\infty} \int_{-\infty}^{\infty} Te^{-\pi q^2} e^{2\pi iqTx} \int_{-\infty}^{\infty} f(z)e^{-2\pi iqTz}dzdq \\
&= \lim_{T\to\infty} \int_{-\infty}^{\infty} Te^{-\pi q^2} e^{2\pi qTx} \hat{f}(Tq)dq \\
&= \lim_{T\to\infty} \int_{-\infty}^{\infty} e^{-\pi \frac{\xi^2}{T^2}} e^{2\pi i\xi x} \hat{f}(\xi)d\xi \\
&= \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} e^{-\pi \frac{\xi^2}{T^2}}\right) \hat{f}(\xi)e^{2\pi i\xi x}d\xi \\
&= \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi x}d\xi.
\end{aligned}$$

□

Example 14. *We computed previously that for* $f(x) = e^{-|x|}$, $\hat{f}(\xi) = \frac{2}{1+4\pi^2\xi^2}$. *This satisfies the conditions of our theorem and so we must have that*

$$f(x) = e^{-|x|} = 2\int_{-\infty}^{\infty} \frac{e^{2\pi i\xi x}d\xi}{1+4\pi^2\xi^2} = 4\int_{0}^{\infty} \frac{\cos(2\pi\xi x)d\xi}{1+4\pi^2\xi^2}.$$

So, for example

$$4\int_0^\infty \frac{1}{1+4\pi^2\xi^2}d\xi = e^0 = 1$$

and

$$4\int_0^\infty \frac{\cos(\pi\xi}{1+4\pi^2\xi^2}d\xi = e^{-\frac{1}{2}}.$$

9.1 Convolution and Plancherel's theorem

Much like for Fourier series, we have a notion of convolution that behaves well with the Fourier transforms

Definition 10. *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be (continuous) absolutely integrable functions. Their convolution is defined by*

$$(f * g)(x) := \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

Fact 2. *Convolution has the following properties*

- $(f * g) = (g * f)$,
- $(f * (g * h)) = ((f * g) * h)$,
- $(f * g)$ *is continuous if at least one of* f *or* g *is continuous*

- *If g is continuously differentiable, then so if $(f * g)$ and $(f * g)' = (f * g')$.*

In particular, the Fourier transform of a convolution is easy to compute in terms of the Fourier transform of the functions involved.

Lemma 9. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be absolutely integrable functions. Then*

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Proof.

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{-\infty}^{\infty} (f * g)(x) e^{-2\pi i \xi x} dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x - y) e^{-2\pi i \xi x} dy dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z) e^{-2\pi i \xi (y+z)} dy dz \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i \xi y} dy \int_{-\infty}^{\infty} g(z) e^{-2\pi i \xi z} dz \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

□

This lets us prove a continuous analogue of Parseval's theorem.

Theorem 6 (Plancherel's Theorem)**.** *Suppose $f : \mathbb{R} \to \mathbb{C}$ is bounded, continuous and with*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

Then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi.$$

Proof. Similarly to our proof of Parseval's theorem, we will consider convolution with the function $\tilde{f}(x) := \overline{f(-x)}$. For purposes of this proof, we will assume f is absolutely integrable, so that the Fourier transform of both f and $\tilde{f}$ exist. Then

$$\begin{aligned}
\hat{\tilde{f}}(\xi) &= \int_{-\infty}^{\infty} \overline{f(-x)} e^{-2\pi i \xi x} dx \\
&= \overline{\int_{-\infty}^{\infty} f(-x) e^{2\pi i \xi x} dx} \\
&= \overline{\int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx} \\
&= \overline{\hat{f}(\xi)}.
\end{aligned}$$

Thus, by the above lemma

$$|\hat{f}(\xi)|^2 = \hat{f}(\xi)\hat{\tilde{f}}(\xi) = \widehat{(f * \tilde{f})}(\xi).$$

Computing the Fourier inverse, we get that

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 e^{2\pi i\xi x} d\xi = \int_{-\infty}^{\infty} \widehat{(f * \tilde{f})}(\xi) e^{2\pi i\xi x} d\xi = (f * \tilde{f})(x).$$

In particular, at $x = 0$, we have that

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = (f * \tilde{f})(0)$$

the right hand side of which is

$$\begin{aligned}(f * \tilde{f})(0) &= \int_{-\infty}^{\infty} f(x)\tilde{f}(-x)dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 dx.\end{aligned}$$

□

Remark 11. *In the above proof, we assumed that f is absolutely integrable, which we need to ensure the Fourier transform exists. However, it is not strictly necessary: Carleson's theorem tells us that if*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

then the limit

$$\lim_{M\to\infty} \int_{-M}^{M} f(x)e^{-2\pi i\xi x} dx$$

exists for almost all ξ. One approach to proving this is to approximate f by absolutely integral functions, but this involves non-trivial arguments.

"There are no easy proofs of Carleson's theorem."

Corollary 8. *If $f, g : \mathbb{R} \to \mathbb{C}$ are bounded and both*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\,, \quad \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty\,,$$

then

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Proof. Note that

$$V := \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$$

is a vector space with a Hermitian positive semidefinite form

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

and associated seminorm

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Plancherel's theorem says that $\|f\|^2 = \|\hat{f}\|^2$. Using the fact that

$$\langle f, g\rangle = \frac{1}{4}\left(\left(\|f+g\|^2 - \|f-g\|^2\right) + i\left(\|f+ig\|^2 - \|f-ig\|^2\right)\right),$$

we conclude that $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, which is precisely the statement of the result. □

Remark 12. *For many of the results discussed, we had to assume that both f and $\hat{f}$ be absolutely integrable. This is not guaranteed. For example*

$$\hat{\chi}_{[0,1]}(\xi) = \frac{e^{2\pi i\xi} - 1}{2\pi i\xi}$$

is not absolutely integrable. The main class of functions for which we can guarantee that absolute integrability for are the Schwartz functions: $f : \mathbb{R} \to \mathbb{C}$ infinitely differentiable such that

$$\sup_{x\in\mathbb{R}} |x^k f^{(l)}(x)| < \infty \quad \text{for all } k, l \geq 0.$$

The Fourier transform defines an automorphism of the space of Schwartz functions.

10 Polynomial approximation and the Weierstrauss theorem

As another application of our Continuous Key Lemma, we will prove the Weierstrauss approximation theorem.

Theorem 7. *Let f be a continuous function on $[a, b]$. Then, for any $\varepsilon > 0$, there exists a polynomial P such that*

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \varepsilon,$$

i.e. f can be uniformly approximated by polynomials.

Proof. Note that by translation and scaling, it suffices to consider continuous functions on $[-1,1]$. We choose a continuous extension g of f to $\mathbb{R}$ such that $g(x)=0$ for all $x \notin [-2,2]$. Let C be a constant such that $|g(x)| < C$ for all $x \in \mathbb{R}$. As g is continuous on a closed bounded interval, we know that it is uniformly continuous and hence

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| g(x) - \int_{-\infty}^{\infty} \rho_T(x-y)g(y)dy \right| = 0$$

for any continuous Dirac family on $(0,\infty)$. Therefore, for all $\varepsilon > 0$, there exists $T > 0$ such that

$$\sup_{x\in\mathbb{R}} \left| g(x) - \int_{-\infty}^{\infty} \rho_T(x-y)g(y)dy \right| < \varepsilon.$$

Taking $\rho_T(x) = Te^{-\pi T^2 x^2}$, we note that the Taylor series expansion

$$\sum_{n=0}^{\infty} T\frac{(-\pi T^2 x^2)^n}{n!}$$

converges uniformly on every closed and bounded interval. Thus, there exists $N > 0$ such that

$$\rho_T(y) - \sum_{n=0}^{N} T\frac{(-\pi T^2 x^2)^n}{n!} < \frac{\varepsilon}{4C}$$

for all $x \in [-4,4]$. Denote by

$$R_N(x,T) := \sum_{n=0}^{N} T\frac{(-\pi T^2 x^2)^n}{n!}.$$

We can therefore conclude that

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \rho_T(x-y)g(y)dy - \int_{-\infty}^{\infty} R_N(x-y,T)g(y)dx \right| \\
=&\left| \int_{-2}^{2} g(y)\left(\rho_T(x-y) - R_N(x-y,T)\right)dy \right| \\
\leq& \int_{-2}^{2} |g(y)|\,|rho_T(x-y) - R_N(x-y,T)|\,dy \\
\leq& 4C \cdot \frac{\varepsilon}{4C} = \varepsilon
\end{aligned}$$

for $x \in [-2,2]$. Note here that we have used that $g(x) = 0$ outside of $[-2,2]$. By the triangle inequality

$$|g(x) - \int_{-\infty}^{\infty} g(y)R_N(x-y,T)dy| \leq 2\varepsilon$$

for all $x \in [-2, 2]$ and hence

$$|g(x) - \int_{-\infty}^{\infty} g(y) R_N(x - y, T) dy| \leq 2\varepsilon$$

for all $x \in [-1, 1]$. Finally note that

$$R_N(x - y, T) = \sum_{n=0}^{2N} r_k(y, T) x^k$$

for some $r_k(y, T)$, and hence

$$\int_{-\infty}^{\infty} g(y) R_N(x - y, T) dy$$

is a polynomial in x. □

Week 5

11 The Fourier transform in higher dimension

We can extend much of the theory of Fourier transforms to functions

$$f : \mathbb{R}^d \to \mathbb{C}$$

with little change from the 1-dimensional case as follows. Denote by $\langle v, w \rangle = v_1 w_1 + \cdots v_d w_d$ the usual inner product on $\mathbb{R}^d$.

Definition 11. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous function such that*

$$\int_{\mathbb{R}^d} |f(x)| dx < \infty$$

where $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$, and $dx := dx_1 dx_2 \ldots dx_n$ is the volume form. The Fourier transform of f is the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx.$$

Many properties of the 1-dimensional Fourier transform still hold, with nearly identical proofs. For example, we have that

$$\frac{\partial \hat{f}}{\partial x_k}(\xi) = 2\pi i \xi_k \hat{f}(\xi)$$

and

$$\frac{\partial \hat{f}}{\partial \xi_k} = -\widehat{2\pi i x_k} f(\xi).$$

Our results relating the parity of f and $\hat{f}$ have a multidimensional analogue in

$$\widehat{f(R-)}(\xi) = \hat{f}(R\xi)$$

for any rotation matrix R.

Example 15. *Suppose we have write $f(x) = f_1(x_1) f_2(x_2) \ldots f_d(x_d)$. Then*

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^d} f_1(x_1) \ldots f_d(x_d) e^{-2\pi i (\xi_1 x_1 + \cdots \xi_d x_d)} dx_1 \ldots dx_d \\
&= \left(\int_{-\infty}^{\infty} f_1(x_1) e^{-2\pi i \xi_1 x_1} dx_1 \right) \cdots \left(\int_{-\infty}^{\infty} f_d(x_d) e^{-2\pi i \xi_d x_d} dx_d \right) \\
&= \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \ldots \hat{f}_d(\xi_d).
\end{aligned}$$

In particular, if $f(x) = e^{-\pi \|x\|^2} = e^{-\pi \left(x_1^2 + \cdots x_d^2\right)}$, then

$$\hat{f}(\xi) = e^{-\pi \xi_1^2} e^{-\pi \xi_2^2} \ldots e^{-\pi \xi_d^2} = e^{-\pi \|\xi\|^2}.$$

Another example is if $f(x) = e^{-(|x_1|+\ldots+|x_d|)}$, where the same argument shows that

$$\hat{f}(\xi) = \prod_{k=1}^{d} \left(\frac{2}{1 + 4\pi^2 \xi_k^2} \right).$$

Of course, not every function splits as a product of functions in each coordinate. This can make computing the Fourier transform of even seemingly simple functions quite challenging. For example, there is no obvious way to compute the Fourier transform of $f(x) = e^{-2\pi\|x\|} = e^{-2\pi\sqrt{x_1^2+\cdots+x_d^2}}$. For this particular example, and a number of similar ones, the subordination trick can come in handy.

Lemma 10. *For all $y \in \mathbb{R}$*

$$e^{-|y|} = \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\frac{y^2}{4u}} dx.$$

Proof. Both sides of the above expression define functions for which the Fourier inversion theorem applies, so it suffices to show that they have equal Fourier transform. The Fourier transform of the left hand side is $\frac{2}{1+4\pi^2\xi^2}$, while the Fourier transform of the right hand side is

$$\begin{aligned}
\int_{-\infty}^\infty \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\frac{y^2}{4u}} e^{-2\pi i \xi y} du\, dy &= \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} \left(\int_{-\infty}^\infty e^{-\frac{y^2}{4u}} e^{-2\pi i \xi y} dy \right) du \\
&= \int_0^\infty 2e^{-u} e^{-4\pi^2\xi^2 u} du \\
&= 2\int_0^\infty e^{-u\left(1+4\pi^2\xi^2\right)} du \\
&= \frac{2}{1+4\pi^2\xi^2}.
\end{aligned}$$

□

We can apply this to $y = \|x\|$ as follows.

Example 16. *Let $f(x) = e^{-2\pi\|x\|}$. Taking the Fourier transform and applying the subordination trick we find*

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi\|x\|} e^{-2\pi i \langle \xi, x \rangle} dx \\
&= \int_{\mathbb{R}^d} \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\frac{\pi^2\|x\|^2}{u}} e^{-2\pi i \langle \xi, x \rangle} du\, dx \\
&= \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} \int_{\mathbb{R}^d} e^{-\frac{\pi^2\|x\|^2}{u}} e^{-2\pi i \langle \xi, x \rangle} dx\, du \\
&= \int_0^\infty \pi^{\frac{d-1}{2}} u^{\frac{d-1}{2}} e^{-u(1+\|\xi\|^2)} du
\end{aligned}$$

which we can compute explicitly be repeated integration by parts to get that

$$\hat{f}(\xi) = C_d \left(1 + \|\xi\|^2\right)^{-\frac{d+1}{2}}$$

for some explicit constant C_d.

11.1 Fourier inversion and Plancherel's theorem

Knowing that the multidimensional Gaussian is its own Fourier transform, we can show that $\rho_T(y) = T^d e^{-\pi T^2 \|y\|^2}$ is a Dirac family. Using a multidimensional version of our Key lemma, we can prove a Fourier inversion theorem almost identically to the 1-dimensional case

Theorem 8. *If $f : \mathbb{R}^d \to \mathbb{C}$ is bounded and continuous, and both*

$$\int_{\mathbb{R}^d} |f(x)| dx < \infty, \quad \int_{\mathbb{R}^d} |\hat{f}(\xi)| d\xi < \infty,$$

then

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

We can define a higher dimensional version of convolution, again satisfying most of the properties of the one dimensional case

Definition 12. *Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be absolutely integrable. Their convolution is defined as*

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x - y) dy.$$

As in the one dimensional case

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

and so we can reproduce the proof of Plancherel's theorem.

Theorem 9. *For $f : \mathbb{R}^d \to \mathbb{C}$ an absolutely integrable function with absolutely integrable Fourier transform*

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi.$$

11.2 Radial functions

Definition 13. *A function $f : \mathbb{R}^d \to \mathbb{C}$ is called radial if either of the equivalent conditions given hold*

- *There exists $f_0 : \mathbb{R} \to \mathbb{C}$ such that $f(x) = f_0(\|x\|)$,*
- *For every rotation R $f(Rx) = f(x)$.*

Note that, as $\hat{f}(R\xi) = \widehat{f(R-)}(\xi)$, the Fourier transform of a radial function is again radial. Computing the Fourier transform of a radical function is often much simpler than that of a general function, as we can sometimes reduce the computation to a single integral using hyperspherical coordinates - in particular, $\hat{f}(0)$ can always be reduced to a single integral multiplied by a constant related to the volume of the unit d-ball. However, for non-zero ξ, computing the integral over the angular coordinates can be tricky, involving special functions such as spherical harmonics.

In the case where $d = 2$, we can be more precise about the form of the Fourier transform. Suppose $f : \mathbb{R}^2 \to \mathbb{C}$ is radial, with $f(x) = f_0(\|x\|)$. Then

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f_0(\|x\|)e^{-2\pi i\langle \xi, x\rangle}dx$$

$$= \int_0^\infty rf_0(r)\int_0^{2\pi} e^{-2\pi ir(\xi_1\cos(\theta)+\xi_2\sin(\theta))}d\theta\, dr.$$

There exist θ_ξ such that $\xi_1\cos(\theta)+\xi_2\sin(\theta) = \|\xi\|\sin(\theta+\theta_\xi$. Since the θ-integral is invariant under translation, we therefore have

$$\hat{f}(\xi) = \int_0^\infty rf_0(r)\int_0^{2\pi} e^{-2\pi ir\|\xi\|\sin(\theta)}d\theta\, dr.$$

Recall from exercise sheet we defined the Bessel functions $J_n(t)$ as the Fourier coefficients of $e^{it\sin(2\pi x)}$. By a change of variables, our θ-integral can be seen to be equal to $2\pi J_0(2\pi r\|xi\|)$, and so

$$\hat{f}(\xi) = 2\pi\int_0^\infty rf_0(r)J_0(2\pi r\|\xi\|)dr.$$

Not only have we reduced our integral to a single integral in terms of mostly well understood functions, unlike e^{ix}, the Bessel function has that $J_0(x) \to 0$ as $x \in \infty$ reasonably quickly. As such, for radial functions, we can define $\hat{f}(\xi)$ for f decays slower than we would need otherwise. The faster decay of $J_0(x)$ also ensure that we can better approximate $\hat{f}(\xi)$ numerically.

12 Solving partial differential equations using the Fourier transform

We have seen previously that Fourier series are a valuable tool for solving partial differential equations on a bounded integral - assuming everything behaves sufficiently well, we can reduce solving the partial differential equation to solving a family of ordinary differential equations and express our solution in terms of a sine or cosine series.

The Fourier transform similarly allows us to reduce partial differential equations, possibly in many variables, to solving a family of ordinary differential equations, though here we in general obtain our solution as an integral.

12.0.1 The heat equation in one dimension

Consider an infinite metal rod with and absolutely integrable initial heat distribution $f(x)$. Let

$$u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$$

be a sufficiently differentiable function such that the heat distribution at time t is given by $u(x,t)$. The function u must satisfy (up to scaling one side by a constant)

$$\begin{gathered} \partial_t u(x,t) = \partial_x^2 u(x,t) \quad \text{ for all } x \in \mathbb{R},\, t > 0 \\ u(x,0) = f(x). \end{gathered}$$

Let us take the Fourier transform of the above differential equation. For u sufficiently well behaved, differentiation with respect to t commutes with taking the Fourier transform in x. Thus

$$\partial_t \hat{u}(\xi,t) = -4\pi^2\xi^2 \hat{u}(\xi,t)$$

for all $\xi \in \mathbb{R}$ and $t > 0$. For each fixed $\xi \in \mathbb{R}$, this gives us an ordinary differential equation in t with solution

$$\hat{u}(\xi,t) = A(\xi)e^{-4\pi^2\xi^2 t}$$

for some $A(\xi)$.

Since we want to find u such that $u(x,0) = f(x)$, we must have that

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$

and hence $A(\xi) = \hat{f}(\xi)$. Thus, a solution to our differential equation is given by the inverse Fourier transform

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-4\pi^2\xi^2 t}e^{-2\pi i\xi x}d\xi.$$

Furthermore, this solution is unique: suppose u_1 and u_2 solve the heat equation with initial condition

$$u_1(x,0) = u_2(x,0) = f(x)$$

Then $v := u_1 - u_2$ is a solution to

$$\begin{gathered} \partial_t v(x,t) = \partial_x^2 v(x,t) \quad \text{ for all } x \in \mathbb{R},\, t > 0 \\ v(x,0) = 0 \end{gathered}$$

and so, as above, the Fourier transform of v is of the form

$$\hat{v}(\xi,t) = A(\xi)e^{-4\pi^2\xi^2 t}$$

and

$$A(\xi) = \hat{v}(\xi,0) = \hat{0} = 0.$$

Thus $\hat{v} = 0$ and so $v = 0$.

Note that we can view

$$\hat{f}(\xi)e^{-4\pi^2\xi^2 t}$$

as the product of $\hat{f}(\xi)$ and $\hat{g}(\xi)$ for $g(x) = \frac{1}{2\pi\sqrt{t}}e^{-\frac{x^4}{4t}}$, and thus

$$\hat{u}(\xi, t) = \widehat{(f * g)}(\xi).$$

As such, we can give an alternative integral representation of $u(x, t)$ via convolution

$$u(x, t) = \frac{1}{2\pi\sqrt{t}} \int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4t}}\, dy.$$

12.0.2 The wave equation in one space dimension

Consider the wave equation in one space dimension - this can be thought of as describing the vibration of an infinite string, or of a light wave. We will assume the functions $f, g : \mathbb{R} \to \mathbb{R}$ describing the initial position and velocity are absolutely integrable with absolutely integrable Fourier transform. Let the amplitude of our wave at a point $x \in \mathbb{R}$ and time $t \in \mathbb{R}_{\geq 0}$ be described by a function

$$u : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}.$$

The function u must satisfy (up to scaling one side by a constant)

$$\begin{aligned} \partial_t^2 u(x, t) = \partial_x^2 u(x, t) \quad &\text{for all } x \in \mathbb{R},\ t > 0, \\ u(x, 0) = f(x)&, \\ \partial_t u(x, 0) = g(x)&. \end{aligned}$$

Taking the Fourier transform with respect to x, the wave equation becomes

$$\partial_t^2 \hat{u}(\xi, t) = -4\pi^2\xi^2\hat{u}(\xi, t).$$

We can solve this for each fixed $\xi \in \mathbb{R}$ to determine that

$$\hat{u}(\xi, t) = A(\xi)e^{2\pi i\xi t} + B(\xi)e^{-2\pi i\xi t}$$

for some functions A, B. The Fourier transform of our initial conditions give that

$$\begin{aligned} \hat{u}(\xi, 0) &= \hat{f}(\xi) \\ \partial_t\hat{u}(\xi, 0) &= \hat{g}(\xi) \end{aligned}$$

and hence

$$\hat{u}(\xi, t) = \frac{1}{2}\left(\hat{f}(\xi) + \hat{G}(\xi)\right)e^{2\pi i\xi t} + \frac{1}{2}\left(\hat{f}(\xi) - \hat{G}(\xi)\right)e^{-2\pi i\xi t}$$

where $G(x)$ is any absolutely integrable antiderivative of g.

The inverse Fourier transform of this is particularly easy to compute and thus we conclude that

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{1}{2}\left(\hat{f}(\xi)+\hat{G}(\xi)\right)e^{2\pi i\xi t+2\pi i\xi x} + \frac{1}{2}\left(\hat{f}(\xi)-\hat{G}(\xi)\right)e^{-2\pi i\xi t+2\pi i\xi x}d\xi \\ &= \frac{1}{2}\left(f(x+t)+G(x+t)+f(x-t)-G(x-t)\right). \end{aligned}$$

The d'Alembert method says that every solution to the wave equation should have the form $u(x,t) = P(x+t) + Q(x-t)$, and we have given an explicit such presentation!

What if we wanted to present this as a convolution integral instead? This is not possible in terms of functions, as there is no function with Fourier transform $e^{2\pi i\xi t}$. However, for a more generalised notion of functions, it is possible.

Definition 14. *A distribution is a linear function from a space of "nice" functions to $\mathbb{C}$ that is continuous for an appropriate topology.*

Example 17. *If $f : \mathbb{R} \to \mathbb{C}$ is of moderate decay, then*

$$h(x) \mapsto \int_{-\infty}^{\infty} f(x)h(x)dx$$

is a distribution.

It is common to write distributions in this form: a distribution A is given as

$$A(h) = \int_{-\infty}^{\infty} a(x)h(x)dx$$

as though there were a function a inducing the map A. We usually identify A and $a(x)$. It is often convenient to think of distributions as generalised functions. Distributions have Fourier transforms in the space of distributions. The integration by parts formula lets us define a notion of the derivative of a distribution. These facts let us extend many results about Fourier transforms to a larger space of functions by allowing the output to be a distribution.

The most important examples of distributions are the Dirac delta distributions $\delta(x-a)$ which are defined by

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx := f(a)$$

The Fourier transform of $\delta(x-a)$ is

$$\int_{-\infty}^{\infty} \delta(x-a)e^{-2\pi i\xi x}dx = e^{-2\pi i\xi a}.$$

Using this, we can write each of the terms in $\hat{u}(\xi,t)$ as the Fourier transform of a function with the a Dirac delta and we find that

$$\begin{aligned}u(x,t) &= \frac{1}{2}\left((f*\delta(z+t))(x) + (G*\delta(z+t))(x) + (f*\delta(z-t))(x) - (G*\delta(z-t))(x)\right)\\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(y)\delta(x+t-y) + G(y)\delta(x+t-y) + f(y)\delta(x-t-y) - G(y)\delta(x-t-y)dy\right)\\ &= \frac{1}{2}\left(f(x+t) + G(x+t) + f(x-t) - G(x-t)\right).\end{aligned}$$

as expected!

12.0.3 The damped wave equation in one space dimension

Consider the initial value problem

$$\begin{aligned}\partial_t^2 u + \partial_t u &= \partial_x^2 u\\ u(x,0) &= f(x)\\ \partial_t u(x,0) &= g(x)\end{aligned}$$

for absolutely integrable $f, g : \mathbb{R} \to \mathbb{C}$ with absolutely integrable Fourier transforms. Taking the Fourier transform, we see that we want to solve

$$\begin{aligned}\partial_t^2 \hat{u}(\xi,t) + \partial_t \hat{u}(\xi,t) + 4\pi^2\xi^2\hat{u}(\xi,t) &= 0\\ \hat{u}(\xi,0) &= \hat{f}(\xi)\\ \partial_t \hat{u}(\xi,0) &= \hat{g}(\xi).\end{aligned}$$

For each ξ, the characteristic polynomial of the ordinary differential equation we get is

$$z^2 + z + 4\pi^2\xi^2 = 0.$$

Denote the roots of this polynomial by $\alpha(\xi)$, $\beta(\xi)$. We have that $\alpha(\xi) = \beta(\xi)$ if and only if $1 - 16\pi^4\xi^4 = 0$, or $\xi = \frac{i^k}{2\pi}$ for $k \in \{0,1,2,3\}$. Away from these points, only two of which are real, the characteristic polynomial has distinct roots and so the general solution is of the form

$$\hat{u}(\xi) = A(\xi)e^{\alpha(\xi)t} + B(\xi)e^{\beta(\xi)t}.$$

Our initial conditions allow us to uniquely determine $A(\xi)$ and $B(\xi)$ in terms of $\hat{f}(\xi)$ and $\hat{g}(\xi)$. As the (inverse) Fourier transform only depends on almost-everywhere behaviour of the function, we can take the inverse Fourier transform of $\hat{u}$ without worrying about what happens at $\xi = \pm(2\pi)^{-1}$ to get that

$$u(x,t) = \int_{-\infty}^{\infty} A(\xi)e^{\alpha(\xi)t + 2\pi i\xi x} + B(\xi)e^{\beta(\xi)t+2\pi i\xi x}d\xi$$

which is the best we can do unless α and β are particularly well behaved.

12.0.4 The wave equation in two space dimensions

The wave equation in two dimensions can be thought of as describing the vibrations of an infinite drum, or of a light wave, with an initial position and velocity. We will assume that the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ describing the initial position and velocity are absolutely integrable. Let the amplitude of our wave at a point $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}_{\geq 0}$ be described by a function

$$u : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}.$$

The function u must satisfy (up to scaling one side by a constant)

$$\begin{gathered}\partial_t^2 u(x,t) = \partial_{x_1}^2 u(x,t) + \partial_{x_2}^2 \quad \text{for all } x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x).\end{gathered}$$

We can take the Fourier transform with respect to the x variables to find that the function $\hat{u}(\xi, t)$ satisfies

$$\partial_t^2 \hat{u}(\xi,t) = -4\pi^2 \xi_1^2 \hat{u}(\xi,t) - 4\pi^2 \xi_2^2 \hat{u}(\xi,t) = -4\pi^2 \|\xi\|^2 \hat{u}(\xi,t).$$

For each fixed $\xi \in \mathbb{R}^2$, this gives us an ordinary differential equation in t that we can solve to find

$$\hat{u}(\xi,t) = A(\xi)\cos(2\pi\|\xi\|t) + B(\xi)\sin(2\pi\|\xi\|t).$$

Taking the Fourier transform of our initial conditions, we must have that

$$\begin{gathered}\hat{u}(\xi,0) = \hat{f}(\xi), \\ \partial_t \hat{u}(\xi,0) = \hat{g}(\xi).\end{gathered}$$

Thus, we must have that

$$\begin{aligned}A(\xi) &= \hat{f}(\xi) \\ B(\xi) &= \frac{\hat{g}(\xi)}{2\pi\|\xi\|}\end{aligned}$$

and so

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi\|\xi\|t) + \hat{g}(\xi)\frac{\sin(2\pi\|\xi\|)}{2\pi\|\xi\|}.$$

Taking the inverse Fourier transform, we find that

$$u(x,t) = \int_{\mathbb{R}^2} \left(\hat{f}(\xi)\cos(2\pi\|\xi\|t) + \hat{g}(\xi)\frac{\sin(2\pi\|\xi\|)}{2\pi\|\xi\|} \right) e^{2\pi i \langle \xi, x \rangle} d\xi$$

is a solution to our differential equation. As in the case of the heat equation, this is the unique solution.

Unlike in the case of the heat equation, we cannot easily rewrite this in terms of a convolution of functions. Even determining an expression in terms of distributions is challenging.

Week 6

13 The Laplace transform

While the Fourier transform is an extremely powerful tool, working in the space of functions for which it is defined can be quite restrictive, particularly if we want the Fourier inversion theorem to hold. If, however, we extend the domain of definition of the Fourier transform to the complex plane, then the integral

$$\hat{f}\left(\frac{s}{2\pi i}\right) = \int_{-\infty}^{\infty} f(x)e^{-sx}dx \quad s \in \mathbb{R}$$

makes sense for a much wider class of functions: functions that grow at most exponentially as $x \to \infty$ and that decay rapidly as $x \to \infty$. This integral is class the bilateral Laplace transform. If $f(x) = 0$ for all $x < 0$ (or all $x < A$ for some $A \in \mathbb{R}$), then we don't have to worry about the decay condition, just about the growth condition. This will be the set up for the (one sided) Laplace transform.

Definition 15. *A function $f : \mathbb{R} \to \mathbb{C}$ is called right-sided if $f(x) = 0$ for all $x < 0$. A right sided function is called a-exponentially bounded if $\sup_{x \geq 0} |f(x)|e^{-ax} < \infty$ for some $a \in \mathbb{R}$. We say f is a-exponentially integrable if f is b-exponentially bounded for all $b > a$.*

Lemma 11. *Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous, right sided, and a-exponentially integrable function. Then*

$$\int_0^{\infty} f(x)e^{-sx}dx$$

exists for all $s \in \mathbb{C}$ with $\Re(s) > a$.

Proof. Since f is piecewise continuous, $f(x)e^{-sx}$ is piecewise continuous and it suffices to show that

$$\int_0^{\infty} |f(x)e^{-sx}|dx < \infty$$

for all $b = \Re(s) > a$. As f is a-exponentially integrable, there exists $C > 0$ such that $|f(x)| < Ce^{ax}$ for all $x \geq 0$. Hence

$$\int_0^{\infty} |f(x)e^{-sx}|dx \leq C \int_0^{\infty} e^{(a-b)x}dx = \frac{C}{b-a}$$

when $b = \Re(s) > a$. □

Definition 16. *For $f : \mathbb{R} \to \mathbb{C}$ a piecewise continuous, right sided, and a-exponentially integrable function, the (one sided) Laplace transform of f is $\mathcal{L}f : \mathbb{C}_{>a} \to \mathbb{C}$,*

$$\mathcal{L}f(s) := \int_0^{\infty} f(x)e^{-sx}dx.$$

where

$$\mathbb{C}_{>a} = \{s \in \mathbb{C} \mid \Re(s) > a\}.$$

As noted earlier, the Laplace transform may be related to the Fourier transform when $a \in \mathbb{R}$. More broadly, for f a piecewise continuous, right sided and a-exponentially integrable function, define $f_b : \mathbb{R} \to \mathbb{C}$ for every real $b > a$ by

$$f_b(x) = f(x)e^{-bx}$$

Then

$$\hat{f}_b(\xi) = \int_{-\infty}^{\infty} f(x)e^{-bx-2\pi i\xi x}dx = \int_{0}^{\infty} f(x)e^{-bx-2\pi i\xi x}dx = \mathcal{L}f(b + 2\pi i\xi)$$

for all $b > a$ and every $\xi \in \mathbb{R}$. As such, we can transfer a number of results from the Fourier transform to the setting of the Laplace transform.

Theorem 10. *Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous, right sided, a-exponentially integrable function and suppose that $\mathcal{L}f(b+2\pi i\xi)$ is an absolutely integrable function of ξ for some fixed $b > a$. Then*

$$f(x) = \int_{-\infty}^{\infty} \mathcal{L}f(b + 2\pi i\xi)e^{(b+2\pi i\xi)x}d\xi$$

for all points of continuity $x \in \mathbb{R}$.

Proof. By (a slightly more general version of) the Fourier inversion theorem, if $\mathcal{L}f(b + 2\pi i\xi) = \hat{f}_b(\xi)$ is absolutely integrable as a function of ξ, then as all points of continuity $x \in \mathbb{R}$ of $f_b(x)$,

$$f(x)e^{-bx} = f_b(x) = \int_{-\infty}^{\infty} \hat{f}_b(\xi)e^{2\pi i\xi x}d\xi = \int_{-\infty}^{\infty} \mathcal{L}f(b + 2\pi i\xi)e^{2\pi i\xi x}d\xi$$

and so

$$f(x) = \int_{-\infty}^{\infty} \mathcal{L}f(b + 2\pi i\xi)e^{(b+2\pi i\xi)x}d\xi$$

at all points of continuity of $f_b(x)$. However, as e^{-bx} is continuous and non-zero, $f(x)$ is continuous wherever $f_b(x)$ is continuous. □

Remark 13. *In order to simplify notation, the inverse Laplace transform given in the above theorem is often written as*

$$f(x) = \int_{b-2\pi i\infty}^{b+2\pi i\infty} \mathcal{L}f(s)e^{sx}ds$$

Example 18. *Let $u : \mathbb{R} \to \mathbb{C}$ denote the Heaviside step function*

$$u(x) := \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

This is right sided, piecewise continuous, and a-exponentially integrable for all $a \geq 0$. The Laplace transform is given by

$$\mathcal{L}u(s) = \int_0^\infty e^{-sx} dx = \frac{1}{s}$$

which is defined for all s with $\Re(s) > 0$.

Note that if any $f : \mathbb{R} \to \mathbb{C}$ can be made right sided by multiplication by $u(x)$. As such, we may often drop the right sided requirement from discussions of the Laplace transform.

Example 19. *Let $f(x) = e^{ax}u(x)$. This is right sided, piecewise continuous and a-exponentially integrable. The Laplace transform is given by*

$$\mathcal{L}f(s) = \int_0^\infty e^{(a-s)x} dx = \frac{1}{s-a}$$

For right sided functions continuous everywhere except maybe at 0, the Laplace inversion theorem tells us that Laplace transforms are invertible. As such, we occasionally determine the inverse Laplace tranform of a rational function using the above example.

Example 20. *Let $f : \mathbb{R} \to \mathbb{C}$ be a right sided, piecewise continuous, exponentially integrable function with Laplace transform*

$$\mathcal{L}f(s) = \frac{1}{s^2 - 3s + 2}.$$

By determining the partial fraction decomposition of $\mathcal{L}f(s)$

$$\frac{1}{s^2 - 3s + 2} = \frac{1}{s-2} - \frac{1}{s-1}$$

we can conclude that

$$f(x) = \left(e^{2x} - e^x\right) u(x)$$

almost everywhere.

As with the Fourier transform, the Laplace transform has a number of useful properties to keep in mind.

Lemma 12. *Let $g : \mathbb{R} \to \mathbb{C}$ be right sided, (twice) continuously differentiable on $[0, \infty)$ and exponentially integrable. Then*

- $f(x) = xg(x) \Rightarrow \mathcal{L}f(s) = -\left(\mathcal{L}g\right)'(s)$,
- $f(x) = g'(x) \Rightarrow \mathcal{L}f(s) = s\mathcal{L}g(s) - g(0)$,
- $f(x) = g''(x) \Rightarrow \mathcal{L}f(s) = s^2\mathcal{L}g(s) - sg(0) - g'(0)$,
- $f(x) = g(x-t)$ *for* $t \geq 0$, $\Rightarrow \mathcal{L}f(s) = e^{-st}\mathcal{L}g(s)$.

Proof. If $f(x) = xg(x)$,

$$\mathcal{L}f(s) = \int_0^\infty xg(x)e^{-sx}dx = -\frac{d}{ds}\int_0^\infty g(x)e^{-sx}dx = -\left(\mathcal{L}g\right)'(s).$$

If $f(x) = g'(x)$,

$$\mathcal{L}f(s) = \int_0^\infty g'(x)e^{-sx}dx = \left[g(x)e^{-sx}\right]_0^\infty + s\int_0^\infty g(x)e^{-sx}dx = -g(0) + \mathcal{L}g(s).$$

The third claim follows similarly by integrating by parts twice. The final claim then follows easily from the definition

$$\mathcal{L}f(s) = \int_0^\infty g(x-t)e^{-sx}dx = \int_{-t}^\infty g(x)e^{-sx-st}dx = e^{-st}\int_0^\infty g(x)e^{-sx}dx = e^{-st}\mathcal{L}g(s).$$

□

Example 21. *A useful example is if $f(x) = xe^{ax}$, then $\mathcal{L}f(s) = \frac{1}{(s-a)^2}$. A more complicated example would be something like*

$$f(x) = xe^{2x}\cos(3x)u(x-2).$$

Letting $g(x) = e^{2x}\cos(3x)u(x-2)$, we find that

$$\mathcal{L}f(s) = -\left(\mathcal{L}g\right)'(s),$$

by the first point of the previous lemma. By the translation rule, we see that mathcalLg$(s) = e^{-2s}\mathcal{L}h(s)$ for $h(x) = e^4e^{2x}\cos(3x+6)u(x)$ so that $g(x) = h(x-2)$. Then

$$\mathcal{L}f(s) = e^{-2s}\left(2\mathcal{L}h(s) - \left(\mathcal{L}h\right)'(s)\right).$$

Thus it suffices to compute $\mathcal{L}h(s)$. We have that

$$h(x) = \frac{e^{4+6i}}{2}e^{2x+3ix}u(x) + \frac{e^{4-6i}}{2}e^{2x-3ix}u(x)$$

the Laplace transform of which is

$$\mathcal{L}h(s) = \frac{e^{4+6i}}{2}\frac{1}{s-2-3i} + \frac{e^{4-6i}}{2}\frac{1}{s-2+3i}$$

and so

$$\mathcal{L}f(s) = \frac{e^{4+6i-2s}}{s-2-3i} + \frac{e^{4-6i-s}}{s-2+3i} + \frac{1}{2}\left(\frac{e^{4+6i-2s}}{(s-2-3i)^2} + \frac{e^{4-6i-2s}}{(s-2+3i)^2}\right).$$

Definition 17. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be piecewise continuous, right sided, exponentially integrable functions. The convolution of f with g is the function $(f * g) : \mathbb{R} \to \mathbb{C}$ is defined by*

$$(f * g)(x) := \int_{-\infty}^\infty f(y)g(x-y)dy = \int_0^x f(y)g(x-y)dy.$$

Lemma 13. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be piecewise continuous, right sided, exponentially continuous functions. Then*

$$\mathcal{L}(f * g)(s) = \mathcal{L}f(s)\mathcal{L}g(s)$$

for all $\Re(s)$ sufficiently large.

Proof. Computing the Laplace transform of the convolution, we see that

$$\begin{aligned}
\mathcal{L}(f * g)(s) &= \int_0^\infty (f * g)(x)e^{-sx}dx \\
&= \int_0^\infty \int_0^x f(y)g(x-y)e^{-sx}dy\,dx \\
&= \int_0^\infty \int_y^\infty f(y)g(x-y)e^{-sx}dx\,dy \\
&= \int_0^\infty \int_0^\infty f(y)g(z)e^{-sz-sy}dz\,dy = \mathcal{L}f(s)\mathcal{L}g(s).
\end{aligned}$$

□

13.1 Solving differential equations with convolution

As with the Fourier transform, we can transform differential equations into algebraic equations using the Laplace transform, but with substantially fewer restrictions on the possible space of solutions.

Example 22. *Suppose $y : [0, \infty) \to \mathbb{C}$ is exponentially integrable and satisfies*

$$y''(x) - 3y'(x) + 2y(x) = 1$$

for all $x > 0$, with initial conditions $y(0) = A$, $y'(0) = B$. Now, as $y(x) = \frac{1}{2}$ is a particular solution to the differential equation, we can use the characteristic equation to obtain a general solution. The initial conditions would then uniquely specify the genuine solution. Instead, we can take the Laplace transform:

$$s^2\mathcal{L}y(s) - sy(0) - y'(0) - 3s\mathcal{L}y(s) + 3y(0) + 2\mathcal{L}y(s) = \frac{1}{s},$$

which reduces to

$$\left(s^2 - 3s + 2\right)\mathcal{L}y(s) = \frac{1}{s} + As + 3A - B$$

and so

$$\mathcal{L}y(s) = \frac{1}{s(s-1)(s-2)} + \frac{As}{(s-1)(s-2)} + \frac{3A - B}{(s-1)(s-2)}.$$

Computing the partial fraction decomposition of the right hand side, we see that

$$\mathcal{L}y(s) = \frac{1}{2}\frac{1}{s} + (B - 1 - 4A)\frac{1}{s-1}\left(5A - B + \frac{1}{2}\right)\frac{1}{s-2}$$

and so we conclude

$$y(x) = \frac{1}{2} + (B - 1 - 4A)\, e^x + \left(5A - B + \frac{1}{2}\right) e^{2x}$$

We can also solve differential equations involving convolutions.

Example 23. *Suppose $y : [0, \infty) \to \mathbb{C}$ is exponentially integrable and satisfies*

$$y'(x) + (g * y)(x) = g(x)$$

for $x > 0$ and $g(x) = e^x u(x)$. Suppose further that $y(0) = 0$. Then, taking the Laplace transform we get

$$s\mathcal{L}y(s) + \mathcal{L}g(s)\mathcal{L}y(s) = \mathcal{L}g(s).$$

As $\mathcal{L}g(s) = \frac{1}{s-1}$, we can solve this for $\mathcal{L}y(s)$ to get

$$\mathcal{L}y(s) = \frac{1}{s-1}\frac{1}{s+\frac{1}{s-1}} = \frac{1}{s^2 - s + 1} = \frac{1}{\sqrt{-3}}\left(\frac{1}{s - \frac{1}{2} - \frac{\sqrt{-3}}{2}} - \frac{1}{s - \frac{1}{2} + \frac{\sqrt{-3}}{2}}\right)$$

and so

$$y(x) = \frac{1}{\sqrt{-3}}\left(e^{\left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)x} - e^{\left(\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)x}\right)$$

13.2 Linear system theory

Many physical processes, such as the result of inputting an electrical impulse through a processor chip of choice, can be described using system theory. As many of the basic results in signal theory rely on complex analytic methods, we shall omit most of the proofs.

Definition 18. *A signal is a function $f : \mathbb{R} \to \mathbb{C}$, assumed to be piecewise continuous.*

A system S is a function from the space of signals to itself: it takes input signal f and produces output signal Sf. A system is called

- *Linear if S is a linear operator,*
- *Time invariant if $ST_t = T_tS$ for all $t \in \mathbb{R}$, where $(T_tf)(x) := f(x - t)$,*
- *Causal if Sf is right sided for every right sided signal f.*

Fact 3. *Linear, time invariant, causal signals can always be expressed as a convolution with a generalised function, called a distribution. We will only consider the case where*

$$Sf = h * f$$

for right sided function h, called the impulse response. (In general, h is given by the output $S\delta$ of the system when given the Dirac delta function as input.)

Definition 19. *A linear, time invariant, casual system S is called exponentially integrable if h is exponentially integrable. We call S stable if Sf is bounded for every bounded f.*

Fact 4. *A system S is stable if and only if the impulse response is absolutely integrable.*

Fact 5. *Suppose a system S is represented by a right sided, exponentially integrable function $h : \mathbb{R} \to \mathbb{C}$ with Laplace transform*

$$H(s) := \mathcal{L}h(s) = \frac{P(s)}{Q(s)}$$

for polynomials P, Q with no common factors and $\deg P < \deg Q$. Then S is stable if and only if the roots of $Q(s)$ have negative real part.

Example 24. *Let S be the system given by $Sf = h * f$ for*

$$h(x) = \frac{1}{2}\cosh(x)u(x) \quad \Rightarrow \quad H(s) = \frac{s}{s^2 - 1}.$$

As the denominator has a root at $s = 1$, the system S is not stable.

In contrast, the system with impulse response

$$h(x) = xe^{-x}u(x) \quad \Rightarrow \quad H(s) = \frac{1}{s^2 + 2s + 1}$$

is stable, as $s^2 + 2s + 1$ has only roots at $s = -1$.

Some systems can be stabilised by upgrading them to systems with feedback

Definition 20. *A system S with constant feedback K is a system T such that*

$$T(f) = S(f + KT(f))$$

for all signals f.

The constant K is often called the amplifier of the system T and S is the base system. Given a pair (S, K) we can always find a system with constant feedback associated to it: suppose S has impulse response h and denote by y the desired output $T(f)$. We will denote by F, H, Y the Laplace transforms of f, h, y respectively. Then we want to determine y such that

$$y = h * (f + Ky)$$

which, assuming we are in a situation where we can invert the Laplace transform, is equivalent to finding Y such that

$$Y = H(F + KY).$$

Hence

$$\mathcal{L}y = Y = \frac{H}{1 - KH}F \quad \Leftrightarrow y = \mathcal{L}^{-1}\left(\frac{H(s)}{1 - KH(s)}\right) * f$$

and the system T is given by convolution with the inverse Laplace transform of $H_K := \left(\frac{H(s)}{1-KH(s)}\right)$.

In the case where $H(s) = \frac{P(s)}{Q(s)}$ discussed above, $H_K(s) = \frac{P(s)}{Q(s)-KP(s)}$. By careful choice of K is is often possible to shift the zeros of $H_K(s)$ to the left and obtain a stable system. Such a modification is could be used in, say for example, a water heating system in order to maintain a stable temperature.

Example 25. *Suppose we have a linear, time invariant, causal system with impulse response*

$$h(x) = \left(\frac{4}{3}e^{x} - \frac{1}{3}e^{-2x}\right) \quad \Rightarrow H(s) = \frac{s+3}{s^2+s-2}.$$

This is not stable - $h(x)$ is not absolutely integrable, and $H(s)$ has a root at $s = 1 > 0$. Let us introduce feedback into the system with amplifier K. The Laplace transform of the modified impulse response has denominator

$$Q(s) - KP(s) = s^2 + (1-K)s - (2+3K)$$

which has roots

$$s_{\pm} = \frac{K-1}{2}\sqrt{\left(\frac{K-1}{2}\right)^2 + 3K + 2}.$$

The square root is imaginary or zero for $K \in [-9,-1]$, and $\frac{K-1}{2} < 0$ if $K < 1$, so any $K \in [-9,-1]$ the modified denominator has roots with negative real part and we get a stable modified system. If $s_{\pm} \in \mathbb{R}$, then $s_- < s_+$, and so it suffices to check that $s_+ < 0$. This is true for $K < -\frac{2}{3}$.

In all of the systems we have considered so far, we have been given the impulse response, so we can usually check stability directly, without this fact. However, in the wild, we often don't have quite so much information about the precise calculation of the output of a system. We might, however, know that the output of a system satisfied some differential equation. By taking the Laplace transform, we can often determine the Laplace transform $H(s)$ of the impulse response, and can therefore deduce facts about the stability of a system even when we cannot compute the inverse Laplace transform explicitly.

Example 26. *Suppose S_λ is a linear, time invariant, casual system such that, for every signal f, the output $y = S_\lambda f$ satisfies*

$$y' + \lambda(a * y) = f.$$

where $a(x) = e^{-x}u(x)$, $\lambda > 0$ and we assume $y(0) = 0$. Taking the Laplace transform of this equation, we obtain that

$$sY(s) - y(0) + \lambda A(s)Y(s) = F(s).$$

As S_λ is obtained by convolution with some right sided, exponentially integrable h, we have that $Y = HF$ and so

$$sH(s)F(s) + \lambda A(s)H(s)F(s) = F(s)$$

and so

$$H(s) = \frac{1}{s + \lambda A(s)}.$$

Note that $A(s) = \frac{1}{s+1}$, so we have

$$H(s) = \frac{s+1}{s^2 + s + \lambda}.$$

While we could compute the inverse Laplace transform of this via partial fractions, we can also just note that the roots of $s^2 + s + \lambda$ are

$$s_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$$

which have negative real part for all $\lambda > 0$. Thus S_λ is stable.

13.3 Solving PDEs via the Laplace transform

Very similarly to the case of the Fourier transform, we can use the Laplace transform to solve certain partial differential equations, up to determining an inverse Laplace transform.

13.3.1 The transport equation

Suppose we have a function $r : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ satisfying

$$\partial_t r(x,t) = -\alpha \partial_x r(x,t) \text{ for all } x, t > 0$$
$$r(0,t) = C, \quad r(x,0) = 0,$$

where α and C are positive real constants. If we assume that $r(x,t)$ is exponentially integrable as a function of t for every fixed x, we can take the Laplace transform, and our problem becomes

$$sR(x,s) - r(x,0) = -\alpha \partial_x R(x,s)$$
$$R(0,s) = \frac{C}{s}, \quad r(x,0) = 0.$$

Treating the first equality as an ODE for each fixed x, we find that

$$R(x,s) = A(s)e^{-\frac{s}{\alpha}x}$$

for some $A(s)$ depending on s. The boundary condition gives us that

$$R(x,s) = \frac{C}{s}e^{-\frac{sx}{\alpha}}.$$

Computing the inverse Laplace transform directly, while not impossible, is challenging. In order to determine r, we instead recall that the Laplace transform of the Heaviside step function $u(x)$ is equal to $\frac{1}{s}$. Using the properties of the Laplace transform, or by direct computation, we find that

$$\mathcal{L}(u(t-a))(s) = \frac{e^{-as}}{s}.$$

Thus, we conclude that

$$r(x,t) = Cu(t - \frac{x}{\alpha})$$

should solve our differential equation. That is to say

$$r(x,t) = \begin{cases} 0 \text{ if } t < \frac{x}{\alpha}, \\ C \text{ if } t \geq \frac{x}{\alpha}. \end{cases}$$

13.3.2 The heat equation

Consider the heat equation on an infinite rod

$$\partial_t r(x,t) = \partial_x^2 r(x,t)$$

with a boundary condition and an initial condition

$$\partial_x r(0,t) = f(t)$$
$$r(x,0) = 0$$

Here, we only wish to find a solution. As such, we can freely impose additional restrictions to help us determine the solution. Let us suppose there exists a solution such that $r(x,t)$ is bounded for each fixed x. This implies both that $r(x,t)$ is exponentially integrable as a function of t for every fixed x, and that

$$\lim_{s\to\infty} |R(x,s)| \leq \lim_{s\to\infty} \int_0^\infty |r(x,t)|e^{-st}dt \leq \lim_{s\to\infty} C_x \int_0^\infty e^{-st}dt = \lim_{s\to\infty} \frac{C_x}{s} = 0.$$

As such, the Laplace transform must decay for each fixed x. Taking the Laplace transform of the heat equation, we find that $R(x,s)$ must satisfy

$$sR(x,s) - r(x,0) = \partial_x^2 R(x,s)$$

and hence

$$R(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x}.$$

Since we must have $\lim_{s\to\infty} R(x,s) = 0$, we need $A(s)$ to decay faster that $e^{-\sqrt{s}x}$ for every $x \in \mathbb{R}$. As such, we might as well take $A(s) = 0$, and so we can take

$$B(s) = -\frac{F(s)}{\sqrt{s}}.$$

Thus

$$R(x,s) = -\frac{F(s)}{\sqrt{s}}e^{-\sqrt{s}x}.$$

Embracing our inner engineers and searching through a table of transforms, we find that $\frac{-e^{-\sqrt{s}x}}{\sqrt{s}}$ is the Laplace transform of

$$\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}}u(t)$$

and thus we can use the convolution property of the Laplace transform to conclude that

$$r(x,t) = \int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{\pi(t-y)}}e^{-\frac{x^2}{4(t-y)}u(t-y)}dy.$$

Since f is assumed to be right sided, this reduces to the finite integral

$$r(x,t) = \int_{0}^{t} f(y)\frac{1}{\sqrt{\pi(t-y)}}e^{-\frac{x^2}{4(t-y)}u(t-y)}dy.$$

Week 7

14 The Radon transform and tomography

Suppose we have a two dimensional body $\mathcal{O}$ of "density" ρ (this will be called the attenuation or absorption constant). If we fire an x-ray from the origin through $\mathcal{O}$ with intensity I_0, and then measure the intensity I on the far side, physics tells us these intensities are related by

$$I = I_0 e^{-d\rho}$$

where d is the distance the ray travels through $\mathcal{O}$.

If we have 2 regions of different attenuation constants ρ_1, ρ_2, we can iterate this rule to determine that the measured intensity will be

$$I = I_0 e^{-d_1\rho_1 - d_2\rho_2}$$

where d_k is the distance traveled through the region with attenuation constant ρ_k, For general object with attenuation $\rho : \mathcal{O} \to \mathbb{R}$, we can imagine chopping $\mathcal{O}$ up into infinitesimally thin layers to conclude

$$I = I_0 e^{\int_L \rho dx}$$

where L is the line along which our x-ray travels, and we extend ρ to a function on $\mathbb{R}^2$ be defining $\rho(x) = 0$ for all $x \notin \mathcal{O}$.

In order to "see" inside $\mathcal{O}$, we can measure I along various lines L to determine a function

$$L \mapsto \int_{:} \rho(x) dx$$

called the Radon transform of ρ. If we can determine ρ from this, we can see the guts of $\mathcal{O}$ - this is possible, but it is easier to work in three dimensions.

14.1 The three dimensional Radon transform

Now consider a function $\rho : \mathbb{R}^3 \to \mathbb{C}$ a function (with compact support). Instead of measuring ρ by firing a single x-ray through $\mathcal{O}$, we will look at the what happens across a plane.

A plane is $\mathbb{R}^3$ is (not uniquely) determined by a unit vector $\gamma \in \mathbb{R}^3$ and a real number t

$$H_{\gamma,t} := \{x \in \mathbb{R}^3 \mid \langle x, \gamma \rangle = t\}.$$

We define the Radon transform of ρ as the function

$$(R\rho)(\gamma, t) := \int_{H_{\gamma,t}} \rho(x) dx.$$

By extending γ to an orthonormal basis (γ, e_1, e_2), we can write the Radon transform as

$$(R\rho)(\gamma, t) = \int_{\mathbb{R}^2} \rho(t\gamma + x_1 e_1 + x_2 e_2) dx_1 dx_2.$$

Given this, can we recover ρ?

14.2 The Radon transform of radial functions

Consider first the case where

$$\rho(x) = \psi(\|x\|^2)$$

for some $\psi : \mathbb{R}_{\geq 0} \to \mathbb{C}$. Then, as (γ, e_1, e_2) are orthonormal

$$\|t\gamma + x_1 e_2 + x_2 e_2\|^2 t^2 + x_1^2 + x_2^2$$

and so, switching to polar coordinates

$$\begin{aligned}
(R\rho)(\gamma, t) &= \int_{\mathbb{R}^2} \psi(t^2 + x_1^2 + x_2^2) dx_1 dx_2 \\
&= 2\pi \int_0^\infty \psi(t^2 + r^2) r dr \\
&= \pi \int_0^\infty \psi(t^2 + u) du.
\end{aligned}$$

Note that this depends only on t^2, so we can find $\phi : \mathbb{R} \to \mathbb{C}$ such that

$$\phi(t^2) = (R\rho)(\gamma, t)$$

and so

$$\begin{aligned}
\phi(t) = \pi \int_0^\infty \psi(t + u) du \\
\pi \int_t^\infty \psi(v) dv..
\end{aligned}$$

Taking the derivative of this, we get that

$$\rho(x) = \psi(\|x\|^2) = -\frac{1}{\pi} \phi'(\|x\|^2).$$

But what about non-radial functions?

14.3 The Fourier slice theorem

Note that the Fourier transform of ρ is given by

$$\begin{aligned}
\hat{\rho}(\eta\gamma) &= \int_{\mathbb{R}^3} \rho(x) e^{-2\pi i \eta \langle x, \gamma \rangle} dx \\
&= \int_{-\infty}^{\infty} \left(\int_{H_{\gamma,t}} \rho(x) dx_1 dx_2 \right) e^{-2\pi i \eta t} dt \\
&= \widehat{(R\rho)}(\gamma, \eta)
\end{aligned}$$

where we take the Fourier transform with respect to t. If we have two bodies with attenuations ρ_1, ρ_2, such that $R\rho_1 = R\rho_2$, this implies $\widehat{(R\rho_1)} = \widehat{(R\rho_2)}$ and so $\hat{\rho}_1 = \hat{\rho}_2$. Thus, if ρ_1, ρ_2 satisfy the conditions for Fourier inversion, this implies $\rho_1 = \rho_2$! Thus, ρ is uniquely determined by its Radon transform.

In order to truly recover ρ, we need to introduce the dual Radon transform. Given

$$F : S^2 \times \mathbb{R} = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \times \mathbb{R} \to \mathbb{C}$$

we define

$$(R^*F)(x) := \int_{S^2} F(\gamma, \langle x, \gamma\rangle) d\sigma(\gamma)$$

or in spherical coordinates

$$(R^*F)(x) = \int_0^\pi \int_0^{2\pi} F\left((\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi), x_1\cos\theta\sin\phi + x_2\sin\theta\sin\phi + x_3\cos\phi\right)\sin\phi \, d\theta d\phi.$$

We also introduce the Laplace operator

$$\Delta f := \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}.$$

In 1917, Radon showed the following result.

Theorem 11. *For $\rho : \mathbb{R}^3 \to \mathbb{C}$ with piecewise continuous with absolutely integrable Fourier transform*

$$\rho(x) = -\frac{1}{8\pi^2}(\Delta R^* R\rho)(x)$$

Proof. We know that

$$\widehat{(R\rho)}(\gamma, \eta) = \hat{\rho}(\eta\gamma).$$

Taking the (one dimensional) inverse Fourier transform with respect to η, we find

$$(R\rho)(\gamma, t) = \int_{-\infty}^{\infty} r\hat{h}o(\eta\gamma)e^{2\pi i\eta t} d\eta.$$

The dual Radon transform of this is

$$(R^*R\rho)(x) = \int_{-\infty}^{\infty}\int_{S^2} \hat{\rho}(\eta\gamma)e^{2\pi i\eta\langle x,\gamma\rangle} d\sigma(\gamma) d\eta.$$

Note that

$$\Delta e^{2\pi i\eta\langle x,\gamma\rangle} = -4\pi^2\eta^2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)e^{2\pi i\eta\langle x,\gamma\rangle} = -4\pi^2\eta^2 e^{2\pi i\eta\langle x,\gamma\rangle}$$

and hence

$$(\Delta R^*R\rho)(x) = -4\pi^2 \int_{-\infty}^{\infty}\int_{S^2} \eta^2\hat{\rho}(\eta\gamma)e^{2\pi i\eta\langle x,\gamma\rangle} d\sigma(\gamma) d\eta$$

which is the spherical coordinate form of the integral

$$(\Delta R^*R\rho)(x) = -8\pi^2 \int_{\mathbb{R}^3} \hat{\rho}(\xi)e^{2\pi i\langle x,\xi\rangle} d\xi = -8\pi^2\rho(x)!$$

□

Example 27. *This is a tough example. but let us consider the Radon transform of $\rho(x) = \chi_B(x)$ of the indicator function of the unit ball. Then*

$$\begin{aligned}(R\rho)(\gamma, t) &= \int_{\mathbb{R}^2} \chi_B(t\gamma + x_1 e_1 + x_2 e_2) dx_1 dx_2 \\ &= \pi(1 - t^2)\chi_{[-1,1]}(t)\end{aligned}$$

is given by the area of the intersection of $H_{\gamma,t}$ with the unit ball. We will now show that we recover the constant function 1 as the inverse Radon transform for $\|x\| < 1$.

$$(R^* R\rho)(x) = \pi \int_{S^2} \left(1 - \langle x, \gamma\rangle^2\right) \chi_{[-1,1]}(\langle x, \gamma\rangle) d\sigma(\gamma).$$

One can check that this is rotationally invariant, so we can freely choose our spherical coordinate system so that $\langle x, \gamma\rangle = \|x\| \cos\phi$. Thus, for $\|x\| \leq 1$

$$\begin{aligned}(R^* R\rho)(x) &= \pi \int_0^\pi \int_0^{2\pi} \left(1 - \|x\|^2 \cos^2\phi\right) \sin\phi \, d\theta d\phi \\ &= 2\pi^2 \int_0^\pi \left(1 - \|x\|^2 \cos^2\phi\right) \sin\phi \, d\phi \\ &= 2\pi^2 (2 - \frac{2}{3}\|x\|^2).\end{aligned}$$

Taking the Laplacian we find

$$-\frac{1}{8\pi^2}(\Delta R^* R\rho)(x) = \frac{1}{6}(2 + 2 + 2) = 1.$$

For $\|x\| > 1$, we need to compute

$$\begin{aligned}(R^* R\rho)(x) =& 2\pi^2 \int_0^\pi \left(1 - \|x\|^2 \cos^2\phi\right) \chi_{[-1,1]}(\|x\| \cos\phi) \sin\phi \, d\phi \\ =& 2\pi^2 \int_0^{\frac{\pi}{2}} \left(1 - \|x\|^2 \cos^2\phi\right) \chi_{[-1,1]}(\|x\| \cos\phi) \sin\phi \, d\phi \\ &+ 2\pi^2 \int_0^{\frac{\pi}{2}} \left(1 - \|x\|^2 \sin^2\phi\right) \chi_{[-1,1]}(\|x\| \sin\phi) \cos\phi \, d\phi \\ =& 4\pi^2 \int_0^1 (1 - \|x\|^2 u^2) \chi_{[-1,1]}(\|x\| u) du \\ =& 4\pi^2 \int_0^{\|x\|^{-1}} (1 - \|x\|^2 u^2) du = \frac{8\pi^2}{3}\|x\|^{-1}.\end{aligned}$$

This is annihilated by the Laplacian. Thus, we recover the indicator function for all non-unit vectors x.

15 The Mellin transform

Given a sufficiently "nice" function $f : \mathbb{R} \to \mathbb{C}$, we can define the Mellin transform of f as

$$\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx.$$

When this converges, it usually only converges in a strip $a < Re(s) < b$ for some real a, b. Nevertheless, we still can obtain a lot of information from the Mellin transform, and it is often used in order to extend the domain of definition of number theoretic functions.

By making the substitution $x = e^-u$, we find that

$$\mathcal{M}f(s) = -\int_{-\infty}^\infty f(e^-u)e^{-su}du = \hat{F}(\frac{s}{2\pi i})$$

where $F(x) = -f(e^{-x})$. As such, we can deduce a number of Fourier theoretic results for the Mellin transform. In particular, we obtain a formula for the inverse Mellin transform: for any $a < c < b$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathcal{M}f(s)ds.$$

Example 28. *Lets consider* $\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}ds$. *This converges for* $Re(s) > 1$, *and defines a special function we cannot compute in general. However, by integration by parts, we can show that*

$$\Gamma(n+1) = n!$$

for all integers $n \geq 1$. *The recursive property of the factorial holds more generally*

$$\Gamma(s+1) = \int_0^\infty e^{-t}t^s dt = s\int_0^\infty e^{-t}t^{s-1}dt = s\Gamma(s).$$

We can use this to extend the definition of $\Gamma(s)$ *to a function on* $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$

$$\Gamma(s) := \begin{cases} \int_0^\infty e^{-t}t^{s-1}dt, & Re(s) > 1 \\ \frac{1}{s}\Gamma(s), & Re(s), \geq 1. \end{cases}$$

This is a fairly common approach to extend sufficiently nice functions on a strip to almost the entire complex plane – we find some sort of functional equation that holds on the domain of definition, and use this equation to define the values everywhere.

Example 29. *Let* $f_z(t) = e^{-zt}$. *The Mellin transform of* $f_z(t)$ *is*

$$\mathcal{M}f_z(s) = \int_0^\infty e^{-zt}t^{s-1}dt = \int_0^\infty z^{-s}e^{-u}u^{s-1}du = \frac{\Gamma(s)}{z^s}.$$

Recall that we defined the Riemann zeta function for Re(s) > 1 by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

Assuming we can swap the order of summation and integration, this implies that

$$\begin{aligned}\Gamma(s)\zeta(s) &= \sum_{n\geq 1} \int_0^\infty e^{-nt} t^{s-1} dt \\ &= \int_0^\infty \sum_{n\geq 1} e^{-nt} t^{s-1} dt \\ &= \int_0^\infty \frac{1}{e^1 - 1} t^{s-1} dt\end{aligned}$$

This integral doesn't converge everywhere, but it does converge close to everywhere, allowing us to define $\zeta(s)$ close to everywhere in $\mathbb{C} \setminus \{1, 0, -1, -2, \ldots\}$. This isn't the best integral representation though - we can find an integral representation that lets us define $\zeta(s)$ for all $s \neq 0, 1$!

Example 30. *Recall that we used Poisson summation to show the following functional equation*

$$\sum_{n\in\mathbb{Z}} e^{\pi t n^2} =: \theta(t) = t^{-\frac{1}{2}} \theta(\frac{1}{t}), \quad t > 0.$$

Let $\psi(t) = \sum_{n\geq 1} e^{-\pi t n^2}$, so that $\theta(t) = 2\psi(t) + 1$. The functional equation then implies that

$$\psi(t) = t^{-\frac{1}{2}} \psi(\frac{1}{t}) + \frac{1}{2} t^{-\frac{1}{2}} - \frac{1}{2}.$$

Consider the Mellin transform of $\psi(t)$:

$$\begin{aligned}\mathcal{M}\psi(\frac{s}{2}) &= \int_0^\infty \sum_{n\geq 1} e^{-\pi t n^2} t^{\frac{s}{2}-1} dt \\ &= \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}-1} dt \\ &= \sum_{n\geq 1} \frac{1}{\pi^{\frac{s}{2}} n^s} \Gamma(\frac{s}{2}) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}\zeta(s)\end{aligned}$$

for Re(s) > 1. But

$$\int_0^\infty \psi(t) t^{\frac{s}{2}-1} dt = \int_0^1 \psi(t) t^{\frac{s}{2}-1} dt + \int \psi(t) t^{\frac{s}{2}-1} dt.$$

The second of these integrals converges for all $s \in \mathbb{C}$. Using the functional equation, the first of these integrals becomes

$$\begin{aligned}\int_0^1 \psi(t)t^{\frac{s}{2}-1}dt &= \int_0^1 t^{-\frac{1}{2}}\psi(\frac{1}{t})t^{\frac{s}{2}-1} + \frac{1}{2}t^{\frac{s}{2}-\frac{3}{2}} - \frac{1}{2}t^{\frac{s}{2}-1}dt \\ &= \int_0^1 \psi(\frac{1}{t})t^{\frac{s}{2}-\frac{3}{2}}dt + \frac{1}{s-1} - \frac{1}{s} \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty t^{-\frac{1}{2}-\frac{s}{2}}\psi(t)dt.\end{aligned}$$

Thus

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(t^{-\frac{1}{2}-\frac{s}{2}} + t^{\frac{s}{2}-1}\right)\psi(t)dt$$

which converges everywhere except for $s = 0$ and $s = 1$.

As $\Gamma(s) \neq 0$ for all s, this implies that we can use this formula to define $\zeta(s)$ for all $s \neq 0, 1$. As $\Gamma(s)$ is infinite for $s = 0, -1, -2, \ldots$, we can in fact extend $\zeta(s)$ to all $s \neq 1$, and can conclude that $\zeta(-2n) = 0$ for all $n > 0$.

Furthermore, the right hand side is invariant under $s \mapsto 1 - s$, so we can deduce the functional equation for the Riemann zeta function

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

15.1 Writing sums as integrals

The Mellin transform can be quite useful for summing certain series, via the inverse Mellin transform, both finite and infinite series. Let us first consider the infinite case.

Suppose $S = \sum_{n\geq 1} f(n)$ for a "nice" function $f : \mathbb{R} \to \mathbb{C}$, with Mellin transform $F(s)$, so that

$$f(x) = \frac{1}{2\pi i}\int_{\text{c-i}\infty}^{\text{c+i}\infty} F(s)x^{-s}ds$$

Assuming F is also well behaved, we can sum this and swap the order of summation and integration to get that

$$S = \frac{1}{2\pi i}\int_{\text{c-i}\infty}^{\text{c+i}\infty} \sum_{n\geq 1} F(s)n^{-s}ds = \frac{1}{2\pi i}\int_{\text{c-i}\infty}^{\text{c+i}\infty} F(s)\zeta(s)$$

which we can often evaluate using contour integration methods from complex analysis.

Example 31. *Consider $f(x) = \frac{\cos(xy)}{x^2}$, with Mellin transform $F(s) = -y^{2-s}\Gamma(s-2)\cos(\frac{\pi s}{2})$ for $2 < Re(s) < 3$. Then for $2 < c < 3$, we have that*

$$S(y) = -\frac{1}{2\pi i}\int_{\text{c-i}\infty}^{\text{c+i}\infty} y^{2-s}\Gamma(s-2)\cos(\frac{\pi s}{2})\zeta(s)ds = -\frac{1}{2\pi i}\int_{\text{c-i}\infty}^{\text{c+i}\infty} y^{2-s}2^{s-1}\pi^s\frac{\zeta(1-s)}{(s-1)(s-2)}ds$$

where we have used a functional equation for both Γ *and* ζ*. Viewing this as a contour integral, we conclude*

$$S(y) = \frac{y^2}{4} - \frac{\pi y}{2} + \frac{\pi^2}{6}.$$

For finite sums, we have Perron's formula.

Theorem 12. *Let* $g(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$ *be a function convergent in some strip* $a < Re(s) < b$*. Then, for all* $x \in \mathbb{R} \setminus \mathbb{Z}$*, we have that*

$$A(x) := \sum_{n\leq x} a(n) = \frac{1}{2\pi i} \int_{\text{c-i}\infty}^{\text{c+i}\infty} \frac{g(s)x^s}{s} ds.$$

Proof. Consider the Mellin transform of $A(x)$:

$$\begin{aligned}
\mathcal{M}A(-s) &= \int_0^\infty A(x)x^{-s-1}dx = \int_1^\infty A(x)x^{-s-1}dx \\
&= \sum_{n\geq 1} \int_n^{n+1} A(n+\frac{1}{2})x^{-s-1}dx = \sum_{n\geq 1} A(n+\frac{1}{2}) \int_n^{n+1} x^{-s-1}dx \\
&= \sum_{n\geq 1} sA(n+\frac{1}{2}) \left(n^{-s} - (n+1)^{-s}\right) \\
&= s\sum_{n\geq 1} \left(A(n+\frac{1}{2}) - A(n-\frac{1}{2})\right) n^{-s} = s\sum_{n\geq 1} \frac{a(n)}{n^s} = sg(s).
\end{aligned}$$

Taking the inverse Mellin transform, we get that

$$A(x) = \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \frac{g(-s)x^s}{s} ds = \frac{1}{2\pi i} \int_{\text{c-i}\infty}^{\text{c+i}\infty} \frac{g(s)x^s}{s} ds.$$

□

Example 32. *For* $a(n) = 1$, $g(s) = \zeta(s)$ *and* $A(x) = \lfloor x \rfloor$*. Thus*

$$\zeta(s) = s\int_0^\infty \frac{\lfloor x \rfloor}{x^{s+1}} dx$$

and

$$\lfloor x \rfloor = \frac{1}{2\pi i} \int_{\text{c-i}\infty}^{\text{c+i}\infty} \frac{\zeta(s)x^s}{s} ds$$

for some $c > 1$ *and all* $x \neq \mathbb{Z}$*.*