Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

MAU34103-1

Science, Technology, Engineering, and Mathematics

School of Mathematics

Mathematics

Junior/Senior Sophister **Michaelmas 2026**

Lie algebras and Lie groups - Sample Exam 2

Date: **Time:**

Dr. Adam Keilthy

Instructions to Candidates:

Attempt **any three questions**. If you attempt all four questions, only your best three will be considered in your grade. Each question is worth 33 points. One point will be awarded for general presentation.

Unless stated otherwise, you may use all statements given in lectures without proof, but must clearly justify that the assumptions of the statement are fulfilled.

PLEASE DO NOT TURN OVER THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

The use of programmable or text-storing calculators is expressly forbidden.

Instructions for Invigilators:

- Log Tables permitted
- Provide blue answer booklets
- Non-Programmable Calculator allowed

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Question 1

i) (10pts) Define a Lie algebra over a field K. Show that, if L_1, L_2 are Lie algebras with Lie brackets $[-,-]_1$, $[-,-]_2$, then $L_1 \oplus L_2$ is a Lie algebra with bracket

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1]_1, [x_2, y_2]_2).$$

ii) (10pts) Let $L = \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}q$ and define

$$[c, f(t)] = 0, \ [f(t), g(t)] = \text{coeff}_{t^{-1}}(f'(t)g(t))q.$$

Show that L is a(n infinite dimensional) Lie algebra.

iii) (6pts) Show $M = \mathbb{C}[t]$ is a Lie subalgebra of L.

iv) (7pts) Show that every non-zero ideal of L contains q, and hence that every non-trivial quotient of L is abelian.

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Question 2

i) (10pts) Show that the unitary group

$$U(n) = \{A \in GL_n(\mathbb{C}) \mid A\overline{A}^T = I\}$$

is a linear group.

ii) (10pts) Determine the Lie algebra $\mathfrak{u}_n$ of the unitary group $U(n)$ and compute its dimension (over $\mathbb{R}$).

iii) (8pts) Consider the Lie group homomorphism

$$\begin{aligned} f : \mathbb{R} &\to U(1) \\ x &\mapsto e^{2\pi i x}. \end{aligned}$$

Determine the induced homomorphism of Lie algebras

$$df : \mathbb{R} \to \mathbb{R}.$$

iv) (5pts) Show that every connected 1-dimensional Lie group G admits a homomorphism $f : \mathbb{R} \to G$, and hence conclude that $G \cong \mathbb{R}$ or $G \cong U(1)$.

Hint: You may use without proof that any non-zero closed subgroup of $\mathbb{R}$ contains a smallest positive element.

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Question 3

i) (10pts) Define a Cartan subalgebra of a semisimple Lie algebra and show that

$$H = \mathbb{C}(E_{1,1} - E_{2,2}) \oplus \mathbb{C}(E_{2,2} - E_{3,3})$$

is a Cartan subalgebra of $L = \mathfrak{sl}_3(\mathbb{C})$.

ii) (8pts) Compute explicitly the restriction of the Killing form to H.

iii) (10pts) Compute the associated root system R of the pair (L, H) and give a polarisation $R = R_+ \sqcup R_-$.

iv) (5pts) Determine a base for the root system given your choice of polarisation. Write all other roots in terms of this base.

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Question 4

i) (5pts) Suppose V is an n-dimensional vector space over $\mathbb{C}$ with an inner product σ. Show there exists an isomorphism between the Lie algebras

$$\mathfrak{o}(\mathbb{C},\sigma) = \{x \in \mathfrak{gl}(V) \mid \sigma(xv,w) + \sigma(v,xw) = 0 \text{ for all} v,w \in V\}$$

and

$$\mathfrak{so}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid x + x^T = 0\}.$$

ii) (8pts) Suppose L is a finite dimensional Lie algebra over a field K with a representation (V,ρ). Show that if V has an inner product σ such that

$$\sigma(\rho(x)v,w) + \sigma(x,\rho(x)w) = 0$$

for all $x \in L$, $\rho(L) \subset \mathfrak{o}(K,\sigma)$.

iii) (7pts) Let $V = \mathfrak{gl}_2(\mathbb{C})$, and $L = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Show that

$$(z_1,z_2)\cdot v := z_1 v + v z_2$$

defines the structure of an L-representation on V.

Here, $z_1 v$ and $v z_2$ are given by multiplication of 2×2-matrices

iv) (5pts) Show that this is a faithful representation.

v) (8pts) Let $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and define a bilinear form on V by

$$\sigma(v,w) = \mathrm{tr}(vQw^TQ).$$

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Show this is an inner product such that

$$\sigma((z_1, z_2) \cdot v, w) + \sigma(v, (z_1, z_2) \cdot w) = 0$$

and hence conclude $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

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