Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

MAU34103-1

Science, Technology, Engineering, and Mathematics

School of Mathematics

Mathematics

Junior/Senior Sophister **Michaelmas 2026**

Lie algebras and Lie groups - Sample Exam 1

Date: **Time:**

Dr. Adam Keilthy

Instructions to Candidates:

Attempt **any three questions**. If you attempt all four questions, only your best three will be considered in your grade. Each question is worth 33 points. One point will be awarded for general presentation.

Unless stated otherwise, you may use all statements given in lectures without proof, but must clearly justify that the assumptions of the statement are fulfilled.

PLEASE DO NOT TURN OVER THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

The use of programmable or text-storing calculators is expressly forbidden.

Instructions for Invigilators:

- Log Tables permitted
- Provide blue answer booklets
- Non-Programmable Calculator allowed

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Question 1

i) (10pts) Define a Lie algebra over a field K. Show that, for a K-vector space V, the set

$$\mathfrak{gl}(V) = \{A : V \to V \mid A \text{ a linear map}\}$$

forms a Lie algebra with bracket $[A, B] = AB - BA$.

ii) (10pts) Show that the space of lower triangular matrices

$$\mathfrak{l} := \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A_{i,j} = 0 \text{ for } j > i\}$$

is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

iii) (7pts) Show that the space of strictly lower triangular matrices

$$\mathfrak{m} := \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A_{i,j} = 0 \text{ for } j \geq i\}$$

is an ideal of $\mathfrak{l}$.

iv) (6pts) Determine the quotient Lie algebra $\mathfrak{l}/\mathfrak{m}$.

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Question 2

i) (8pts) Express the Lie algebra $L(G)$ of a linear group $G \subset GL_n(\mathbb{R})$ using the matrix exponential.

ii) (5pts) Show that $\det(e^A) = e^{\mathrm{tr}(A)}$

iii) (10pts) Determine the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ of the orthogonal Lie group

$$SL_3(\mathbb{R}) = \{A \in GL_3(\mathbb{R}) \mid \det A = 1\}$$

and determine its dimension.

iv) (10pts) Show that the map

$$f(A) := (A^T)^{-1}$$

defines a group automorphism of $SL_3(\mathbb{R})$ and determine the associated Lie algebra endomorphism

$$df : \mathfrak{sl}_3(\mathbb{R}) \to \mathfrak{sl}_3(\mathbb{R}).$$

Do not forget to explain why f is a smooth map.

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Question 3

i) (10pts) Define a representation of a Lie algebra and an irreducible representation of a Lie algebra.

ii) (8pts) Show that the standard representation of $\mathfrak{gl}_n(\mathbb{C})$ is irreducible.

iii) (10pts) Let $V(n)$ be the unique $n+1$ dimensional irreducible representation of $\mathfrak{sl}_2$, and let $V = V(2) \otimes V(2)$ be the 9-dimensional tensor product representation. Determine the decomposition of V as a direct sum of irreducibles.

iv) (5pts) Show that the dual representation $V(n)^*$ is isomorphic to $V(n)$.

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Question 4

i) (8pts) Define a solvable Lie algebra in terms of the derived series

ii) (10pts) Consider the Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid \theta, x, y \in \mathbb{R} \right\}$$

Show that $\mathfrak{g}$ is solvable.

iii) (10pts) Switching to the complexification $\mathfrak{g} \otimes \mathbb{C}$ by allowing complex entries, determine a basis of $\mathbb{C}^3$ in which every matrix in $\mathfrak{g} \otimes \mathbb{C}$ is upper triangular.

iv) (5pts) Hence or otherwise, whether $\mathfrak{g} \otimes \mathbb{C}$ is nilpotent.

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