MAU34103 - Lie algebras and Lie groups

Practice Sheet 3

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

---

1 Representations of Lie algebras

Exercise 1

Recall that a representation of a Lie algebra L is defined as a Lie algebra homomorphism

$$\rho : L \to \mathfrak{gl}(V)$$

to the endomorphisms of a vector space V. The structure of an L-module on V is given by a bilinear map $L \times V \to V$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all $x, y \in L$, $v \in V$.

Convince yourself that an L-representation (V, ρ) and an L-module V are equivalent structures.

Solution 1

Suppose we have an L-module V. As the module action is bilinear, the function $\rho(x) : V \to V$ defined by

$$\rho(x)(v) := x \cdot v \quad \text{for all} \quad v \in V$$

is a linear map, and so defines a map $\rho : L \to \mathfrak{gl}(V)$. Bilinearity also implies ρ is linear. To see that it is a Lie algebra homomorphism, we note that

$$\rho([x,y])(v) = [x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) = \rho(x)(\rho(y)(v)) - \rho(y)(\rho(x(v))$$

and hence

$$\rho([x,y]) = [\rho(x), \rho(y)].$$

Therefore (V, ρ) is an L-representation. Conversely, given an L-representation (V, ρ) defining

$$x \cdot v := \rho(x)(v)$$

gives the structure of an L-module. Linearity of ρ and of $\rho(x)$ implies bilinearity, and since

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

the commutator condition is satisfied.

Exercise 2

Define the adjoint representation of a Lie algebra L by the map

$$\rho : L \to \mathfrak{gl}(L)$$

where $\rho(x)(y) := [x,y]$ for all $x, y \in L$.

i) Verify that this is indeed a Lie algebra representation, i.e. that ρ is a Lie algebra homomorphism.

ii) In the case of $L = \mathfrak{sl}_2(\mathbb{C})$, determine the matrices of $\rho(x), \rho(y), \rho(h)$ in the basis $\{x, y, h\}$ of L.

iii) Verify explicitly that these matrices satisfy

$$[\rho(u), \rho(v)] = \rho([u,v])$$

for all $u, v \in \{x, y, h\}$.

Solution 2

i) Linearity follows as

$$\rho(ax+y)(z) = [ax+y,z] = a[x,z]+[y,z] = (a\rho(x)+\rho(y))(z)$$

for all $a \in K$, $x, y, z \in L$.

The commutator property follows from the Jacobi identity:

$$\begin{aligned}\rho([x,y])(z) &= [[x,y],z] \\ &= -[[y,z],x] - [[z,x],y] \\ &= [x,[y,z]] - [y,[x,z]] \\ &= \rho(x)(\rho(y)(z)) - \rho(y)(\rho(x)(z)) \\ &= [\rho(x),\rho(y)](z)\end{aligned}$$

for all $x, y, z \in L$.

ii) We compute

$$\begin{aligned}\rho(x)(x) = 0,\ \rho(x)(y) = h,\ \rho(x)(h) = -2x, \\ \rho(y)(x) = -h,\ \rho(y)(y) = 0,\ \rho(y)(h) = 2y, \\ \rho(h)(x) = 2x,\ \rho(h)(y) = -2y,\ \rho(h)(h) = 0,\end{aligned}$$

Thus, the corresponding matrices are

$$\rho(x) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \rho(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix},\ \rho(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

iii) This is just a routine computation.

Exercise 3

Let $L = \mathfrak{sl}_2(\mathbb{C})$ be the special linear Lie algebra and define a representation

$$\rho : L \to \mathfrak{gl}(\mathbb{C}^3)$$

by

$$\begin{aligned}\rho(x)(e_1) = 0,\ \rho(x)(e_2) = e_3,\ \rho(x)(e_3) = -2e_1, \\ \rho(y)(e_1) = -e_3,\ \rho(y)(e_2) = 0,\ \rho(y)(e_3) = 2e_2, \\ \rho(h)(e_1) = 2e_1,\ \rho(h)(e_2) = -2e_2,\ \rho(h)(e_3) = 0.\end{aligned}$$

i) Determine the matrices of $\rho(x), \rho(y), \rho(h)$ in the standard basis

ii) Show that $(\mathbb{C}^3, \rho)$ is isomorphic to the adjoint representation (L, ad).

iii) Classify by hand all homomorphisms of representations

$$f : (\mathbb{C}^3, \rho) \to (L, \mathrm{ad})$$

Hint: We must have that $f(\rho(h)v) = \mathrm{ad}_h(f(v))$. *Consider eigenvectors!*

Solution 3

i) We compute

$$\rho(x) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \rho(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix},\ \rho(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

ii) The map $f : \mathbb{C}^3 \to \mathfrak{sl}_2(\mathbb{C})$ given by

$$e_1 \mapsto x,\ e_2 \mapsto y,\ e_3 \mapsto h$$

satisfies $f(\rho(z)v) = [z, f(v)]$, as the matrices of $\rho(x), \rho(y), \rho(h)$ and $\mathrm{ad}_x, \mathrm{ad}_y, \mathrm{ad}_h$ are equal in the corresponding bases. It is an isomorphism as it is an isomorphism of vector spaces.

iii) Suppose

$$\begin{aligned} f(e_1) &= a_1 x + a_2 y + a_3 h, \\ f(e_2) &= b_1 x + b_2 y + b_3 h, \\ f(e_3) &= c_1 x + c_2 y + c_3 h. \end{aligned}$$

As the action of $\mathfrak{sl}_2$ must commute with f, we must have

$$2f(e_1) = f(2e_1) = f(\rho(h)(e_1) = [h, f(e_1)] = 2a_1 x - 2a_2 y.$$

Hence $a_2 = a_3 = 0$. Similarly, $b_1 = b_3 = 0$. Considering

$$f(e_3) = f(\rho(x)e_2) = [x, f(e_2)] = b_2 h$$

we see that $c_1 = c_2 = 0$ and $c_3 = b_2$. Similarly as

$$f(e_3) = -f(\rho(y)e_1)$$

we find $a_1 = b_2 = c_3$. Thus, every homomorphism between our representations is a scalar multiple of the isomorphism of the previous part.

Exercise 4

Let (V, ρ) be an L-representation over a field K.

i) Let $V^* = \mathrm{Hom}(V, K)$. Define an action of L on V^* by

$$(x \cdot f)(v) := -f(x \cdot v)$$

Show that this defines the structure of an L-representation on V^*.

ii) Denote this dual representation by (V^*, ρ^*). Suppose that $\rho(x) \in \mathfrak{gl}(V)$ is given by a matrix $A(x)$ in some basis of V. Show that $\rho^*(x)$ has matrix $-A(x)^T$ in the dual basis of V^*.

Solution 4

i) It is clear that $x \cdot f \in V^*$ as x acts linearly on V. Linearity of $f :\in V^*$ implies

$$\begin{aligned}((ax + y) \cdot f)(v) &= -f((ax + y) \cdot v) = -f(ax \cdot v + y \cdot v) \\ &= -af(x \cdot v) - f(y \cdot v) = a(x \cdot f)(v) + (y \cdot f)(v)\end{aligned}$$

for all $v \in V$ and so

$$(ax + y) \cdot f = a(x \cdot f) + y \cdot f$$

for all $a \in K$, $x, y \in L$, $f \in f^*$.Thus, we obtain a bilinear map $L \times V \to V$. To see that we get an L-module structure (equivalent to an L-representation), we just need to check the commutator relation:

$$\begin{aligned}([x, y] \cdot f)(v) &= -f([x, y] \cdot v) \\ &= -f(x \cdot y \cdot v - y \cdot x \cdot v) \\ &= -f(x \cdot y \cdot v) + f(y \cdot x \cdot v) \\ &= (x \cdot f)(y \cdot v) - (y \cdot f)(x \cdot v) \\ &= -(y \cdot x \cdot f)(v) + (x \cdot y \cdot f)(v)\end{aligned}$$

for all $v \in V$. Thus

$$[x, y] \cdot f = x \cdot y \cdot f - y \cdot x \cdot f$$

as needed.

ii) Let $v_1, \ldots, v_n$ be the basis of V and $v^1, \ldots, v^n$ be the dual basis. Let the matrix of $\rho^*(x)$ have (i,j)-entry given by $b_{i,j}$, so that

$$\rho^*(x)(v^i) = \sum_{j=1}^{n} b_{i,j} v^j$$

. Then

$$\rho^*(x)(v^i)(v_j) = \sum_k b_{i,k} v^j v_k = b_{i,j}$$

and so

$$b_{i,j} = \rho^*(x)(v^i)(v_j) = -v^i(\rho(x)(v_j)).$$

If $\rho(x)$ has matrix $A(x) = (a_{i,j})$, then

$$b_{i,j} = -v^i(\sum_k a_{j,k} v_k) = -a_{j,i}$$

Thus $\rho^*(x)$ has matrix $-A(x)^T$.

Exercise 5

The vector space $V = \mathbb{C}^2$ can be viewed as a representation of any Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$ via matrix multiplication.

i) View V as a representation of $\mathfrak{gl}_2(\mathbb{C})$. Show V^* is not isomorphic to V as a representation of $\mathfrak{gl}_2(\mathbb{C})$.

ii) View V as a representation of $\mathfrak{sl}_2(\mathbb{C})$. Show V^* *is* isomorphic to V as a representation of $\mathfrak{sl}_2(\mathbb{C})$.

Solution 5

i) Let e_1, e_2 be the standard basis of V and e^1, e^2 be the dual basis. Suppose $f : V \to V^*$ is an isomorphism, with $f(e_1) = ae^1 + be^2$, $f(e_2) = ce^1 + de^2$, and denote the associated matrix by m. Since z acts on V^* by $-z^T$, if f is an homomorphism, we must have

$$-z^T m = mz$$

for all $z \in \mathfrak{gl}_2(\mathbb{C})$. Considering the x running over the basis of $\mathfrak{gl}_2(\mathbb{C})$, we find

$$\left(\begin{smallmatrix} -a & -b \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right)$$

and so on, giving $a = b = c = d = 0$. Thus, we cannot have an isomorphism.

ii) The same condition must hold, but only for the standard basis x, y, h of $\mathfrak{sl}_2(\mathbb{C})$. We obtain the following conditions

$$a = d = 0,\ b = -c.$$

Taking $b = -c = 1$, we obtain the needed isomorphism.

Exercise 6

Let V, W be two L-representations over a field K

i) Show that the action of L on $\mathrm{Hom}(V, W)$ given by

$$(x \cdot f)(v) = x \cdot (f(v)) - f(x \cdot v)$$

defines the structure of an L-representation/module on $\mathrm{Hom}(V, W)$

ii) Show that the map

$$\begin{aligned} V^* \otimes W &\to \mathrm{Hom}(V, W), \\ f \otimes w &\mapsto (v \mapsto f(v)w) \end{aligned}$$

is an isomorphism of L-representations.

Solution 6

i) Bilinearity follows from the linearity of f and the linearity of the action of L on V and W, so only the commutator condition needs to be checked. For all $v \in V$, we have

$$\begin{aligned} ([x, y] \cdot f)(v) &= [x, y] \cdot (f(v)) - f([x, y] \cdot v) \\ &= x \cdot y \cdot f(v) - y \cdot x \cdot f(v) - f(x \cdot y \cdot v) + f(y \cdot x \cdot v) \end{aligned}$$

and

$$\begin{aligned}(x\cdot y\cdot f)(v)-(y\cdot x\cdot f)(v) =& x\cdot((y\cdot f)(v))-(y\cdot f)(x\cdot v)\\ &-y\cdot((x\cdot f)(v))+(x\cdot f)(y\cdot v)\\ =& x\cdot y\cdot f(v)-x\cdot f(y\cdot v)-y\cdot f(x\cdot v)+f(y\cdot x\cdot v)\\ &-y\cdot x\cdot f(v)+y\cdot f(x\cdot v)+x\cdot f(y\cdot v)-f(x\cdot y\cdot v)\\ &=x\cdot y\cdot f(v)-y\cdot x\cdot f(v)-f(x\cdot y\cdot v)+f(y\cdot x\cdot v).\end{aligned}$$

Thus, we obtain a representation of L.

ii) We first note that

$$\dim(V^*\otimes W)=\dim V^*\times\dim W=\dim V\times\dim W=\dim\mathrm{Hom}(V,W)$$

and claim the map is injective. Suppose the map $\phi_{f,w}:V\to W$ given by

$$\phi_{f\otimes w}(v)=f(v)w$$

is the zero map. Either $f(v)=0$ for all $v\in V$ and so $f=0$, or if $f(v)\neq 0$, then $w=0$. In either case $f\otimes w=0$. Thus the map $f\otimes w\to\phi_{f\otimes w}$ is injective. Thus, for dimension reasons, it is an isomorphism of vector spaces.

To see that it is an isomorphism of representations, we must check that it commutes with the action of L. That is, we must have that

$$\phi_{x\cdot(f\otimes w)}(v)=(x\cdot\phi_{f\otimes w})(v)$$

for all $v\in W$, $f\in V^*$, $w\in W$, $x\in L$. Explicitly, we have

$$\begin{aligned}\phi_{x\cdot(f\otimes w}(v) &= \phi_{(x\cdot f)\otimes w+f\otimes x\cdot w}(v)\\ &=\phi_{(x\cdot f)\otimes w}(v)+\phi_{f\otimes x\cdot w}(v)\\ &=(x\cdot f)(v)w+f(v)x\cdot w\\ &=-f(x\cdot v)w+x\cdot(f(v)w)\\ &=x\cdot(\phi_{f\otimes w})(v)\end{aligned}$$

as needed.

2 Solvable and nilpotent Lie algebras

Exercise 7

Let $\mathfrak{b}_3 \subset \mathfrak{gl}_3(\mathbb{C})$ be the Lie algebra of upper triangular matrices.

i) Find the central and derived series of $\mathfrak{b}_3$

ii) Determine if $\mathfrak{b}_3$ is nilpotent, solvable, or neither.

Solution 7

i) A basis of $\mathfrak{b}_3$ is given by $\{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,2}, E_{2,3}, E_{3,3}\}$. We explicitly compute all commutators to check that

$$\mathfrak{b}_3^1 = \mathfrak{b}_3^{(1)} = \mathfrak{n}_3 = \langle E_{1,2}, E_{1,3}, E_{2,3} \rangle$$

and hence

$$\mathfrak{b}_3^{(2)} = \mathbb{C}E_{1,3},\ \mathfrak{b}_3^{(3)} = 0.$$

From the computation of $\mathfrak{b}_3^1$, we find $\mathfrak{b}_3^2 = \mathfrak{b}_3^1$ and so $\mathfrak{b}_3^n = \mathfrak{n}_3$ for all $n \geq 1$.

ii) It is solvable, but not nilpotent.

Exercise 8

Recall that any two-dimensional non-abelian Lie algebra L is isomorphic to the Lie algebra with basis $\{x, y\}$ and bracket determined by $[x, y] = x$. Is this Lie algebra nilpotent, solvable, or neither?

Solution 8

We have that $[L, L] = Kx$, and so $L^{(2)} = 0$, but $L^2 = Kx$. Thus L is solvable, but not nilpotent.

Exercise 9

i) If $f : L \to M$ is a surjective Lie algebra homomorphism, show that $f(\operatorname{rad} L) \subset \operatorname{rad}(M)$,

ii) If L, M are Lie algebras, show that $\operatorname{rad}(L \oplus M) = \operatorname{rad}(L) \oplus \operatorname{rad}(M)$,

iii) If $I \subset L$ is a solvable ideal, show that $\operatorname{rad}(L/I) \cong \operatorname{rad}(L)/I$.

Solution 9

i) Since $f([x, y]) = [f(x), f(y)]$, we must have $f(I^{(k)}) \subset f(I)^{(k)}$ for any ideal $I \subset L$. So if $f(I)$ is an ideal, then I being solvable implies $f(I)$ is solvable. Since $\operatorname{rad} L$ is a solvable ideal, and $\operatorname{rad} M$ contains all solvable ideals, the claim would then follow.

Since f is surjective, then for any $m \in M$, there exists $x \in L$ such that $m = f(x)$. Thus, for any $i \in I$

$$[m, f(i)] = [f(x), f(i)] = f([x, i]) \in f(I)$$

as $[x, i] \in I$. Hence $f(I)$ is an ideal.

ii) Since $[\operatorname{rad} L, \operatorname{rad} M] = 0$, we will have both that $\operatorname{rad} L \oplus \operatorname{rad} M$ is an ideal of $L \oplus M$ and that $(\operatorname{rad} L \oplus \operatorname{rad} M)^{(k)} = (\operatorname{rad} L)^{(k)} \oplus (\operatorname{rad} M)^{(k)}$. Hence it is a solvable ideal and so $\operatorname{rad} L \oplus \operatorname{rad} M \subset \operatorname{rad}(L \oplus M)$.

Next note that for any ideal $I \subset L \oplus M$, we have that $I \cap L$ is an ideal of L. It is a vector subspace, and if $x \in I \cap L$ and $y \in L$, then $[x, y] \in I$ (I is an ideal) and $[x, y] \in L$ (as L is a subalgebra). Furthermore, as

$$(I \cap L)^{(k)} = I^{(k)} \cap L$$

if I is solvable, then $I \cap L$ is solvable. Thus

$$\operatorname{rad}(L \oplus M) \cap L \subset \operatorname{rad} L.$$

Similarly for M. But

$$\operatorname{rad}(L \oplus M) = (\operatorname{rad}(L \oplus M) \cap L) \oplus (\operatorname{rad}(L \oplus M) \cap M) \subset \operatorname{rad} L \oplus \operatorname{rad} M.$$

Therefore we have equality.

iii) Lie π be the natural map $L \to L/I$. As π is a surjective map,

$$\pi(\mathrm{rad}(L)) \subset \mathrm{rad}(L/I).$$

As $I \subset \mathrm{rad}(L)$, $\mathrm{rad}(L)/I = \pi(\mathrm{rad}(L))$. Thus $\mathrm{rad}(L)/I \subset \mathrm{rad}(L/I)$.

Suppose $J \subset L/I$ is a solvable ideal. Then $Q = \pi^{-1}(J)$ is an ideal of L containing I. Since J is solvable, there exists $k > 0$ such that $J^{(k)} = 0$ in L/I. Hence

$$Q^{(k)} = \pi^{-1}(J^{(k)}) \subset I.$$

As I is solvable, there is an $n > 0$ such that $Q^{(k+n)} = 0$ and so Q is solvable. Thus the map $\mathrm{rad}(L) \to \mathrm{rad}(L/I)$ is surjective, and so $\mathrm{rad}(L)/I = \mathrm{rad}(L/I)$.

Exercise 10

Show that $\mathfrak{gl}_n(\mathbb{C})$ is not solvable for any $n \geq 0$ by computing the derived algebra of $\mathfrak{gl}_n(\mathbb{C})$ explicitly. Hence conclude $\mathfrak{sl}_n(\mathbb{C})$ is not solvable.

Solution 10

By computing the commutator of basis elements $[E_{i,j}, E_{k,\ell}]$, we see that

$$[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$$

Since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}I$, we have

$$\mathfrak{sl}_n = [\mathfrak{gl}_n, \mathfrak{gl}_n] =\subset [\mathfrak{sl}_n, \mathfrak{sl}_n] + [\mathfrak{sl}_n, \mathbb{C}I] + [\mathbb{C}I, \mathfrak{sl}_n] + [\mathbb{C}I, \mathbb{C}I] = [\mathfrak{sl}_n, \mathfrak{sl}_n].$$

Thus the derived series is

$$\mathfrak{gl}_n \supset \mathfrak{sl}_n \supset \mathfrak{sl}_n \supset \cdots$$

which never goes to 0. Thus $\mathfrak{gl}_n$ is not solvable, nor is $\mathfrak{sl}_n$.

Alternatively, one can use that $\mathfrak{gl}_n$ is solvable if and only if $[\mathfrak{gl}_n, \mathfrak{gl}_n]$ is nilpotent, and such that there are elements of $\mathfrak{sl}_n$ which are not nilpotent.

Exercise 11

Determine a non-zero Lie algebra $L \subset \mathfrak{gl}_n(\mathbb{C})$ such that L is nilpotent, but L contains a matrix x such that $x^N \neq 0$ for any $N > 0$.

Solution 11

Take any Lie subalgebra containing the identity.

3 Simple and semisimple Lie algebras

Exercise 12

Show directly that $\mathfrak{sl}_2$ is a simple Lie algebra over a field of characteristic 0.

Solution 12

Suppose $\mathfrak{sl}_2$ contains a non-zero ideal I, with a non-zero element $v = ax + by + ch$. If $a \neq 0$, then $[y, v] = -2ah + 2cy \in I$, and so $[y, [y, h]] = 4ay \in I$. Therefore $y \in I$, which implies $[x, y] = h \in I$ and so $\frac{-1}{2}[x, h] = x \in I$. Therefore $I = \mathfrak{sl}_2$. If $b \neq 0$, then

$$[x, [x, v]] = [x, bh - 2cx] = -2bx \in I$$

We then similarly conclude $h \in I$ and $y \in I$. If $a = b = 0$, $c \neq 0$, then $h \in I$ and so $[x, h] = -2x \in I$ and $[y, h] = 2y \in H$. Thus $I = \mathfrak{sl}_2$.

Exercise 13

Define a bilinear form on $\mathfrak{gl}_n(\mathbb{C})$ by

$$(x, y) := \operatorname{tr}(xy)$$

i) Show that $(-, -)$ is non-degenerate,

ii) Show that $\mathfrak{sl}_n(\mathbb{C})$ is orthogonal to $\mathbb{C}\,\mathrm{id}$ with respect to this form,

iii) Show that $(-, -)|_{\mathfrak{sl}_n}$ is non-degenerate and conclude $\mathfrak{sl}_n(\mathbb{C})$ is reductive,

Solution 13

i) If $(E_{i,j}, x) = 0$, then $x_{i,j} = 0$, and so if $(x, y) = (y, x) = 0$ for all $y \in \mathfrak{gl}_n(\mathbb{C})$, $x = 0$. Thus $(-, -)$ is non-degenerate.

ii) We have that $(x, cI) = \operatorname{tr}(x(cI)) = c \operatorname{tr}(x) = 0$ for all $cI \in \mathbb{C}\,\mathrm{id}$ and $x \in \mathfrak{sl}_n(\mathbb{C})$. Thus they are orthogonal.

iii) As in the first part, we must have that if $(x, y) = 0$ for all $y \in \mathfrak{sl}_n$, then $x_{i,j} = 0$ for all $i \neq j$, looking at $(x, E_{i,j})$. Considering $(x, E_{i,i} - E_{i+1,i+1})$, we see that $x_{i,i} = x_{i+1,i+1}$ for all $1 \leq i \leq n-1$. Thus

$$0 = \operatorname{tr} x = x_{1,1} + \cdots + x_{n,n} = x_{1,1} + \cdots + x_{1,1} = nx_{1,1}$$

and so $x_{i,i} = x_{1,1} = 0$ for all i. Therefore $x = 0$.

Exercise 14

Using that L is reductive if it admits a faithful representation (V, ρ) such that the bilinear form

$$(x, y) := \operatorname{tr}_V(\rho(x)\rho(y)$$

is non-degenerate, show that $\mathfrak{so}_n(K)$, $\mathfrak{sp}_{2n}$, and $\mathfrak{su}(n)$ are semisimple.

Solution 14

We consider the standard representation of each. A basis of $\mathfrak{so}_n(K)$ is given by $\{E_{i,j} - E_{j,i}\}_{i<j}$, and

$$\operatorname{tr}((E_{i,j} - E_{j,i})(E_{k,\ell} - E_{\ell,k})) = \begin{cases} -2 \text{ if } (i,j) = (k,\ell), \\ 0 \text{ otherwise.} \end{cases}$$

As such, the matrix of $(-,-)$ for $\mathfrak{so}_n(K)$ is $-2\,\mathrm{id}$ which is invertible and so $(-,-)$ is non-degenerate (assuming we are in characteristic different to 2). Thus $\mathfrak{so}_n$ is reductive and a subalgebra of $\mathfrak{sl}_n$, so semisimple.

Since both $\mathfrak{sp}_{2n}$ and $\mathfrak{su}(n)$ are subalgebras of the special linear Lie algebra, it is similarly enough to show the given bilinear form is non-zero for the standard representation of $\mathfrak{sp}_{2n}$ and the standard (real) representation of $\mathfrak{su}(n)$.

The symplectic Lie algebra consists of matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

where $B = B^T$ and $C = C^T$. The product of two such matrices is

$$\begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 - B_1A_2^T \\ C_1A_2 - A_1^TC_2 & C_1B_2 + A_1^TA_2^T \end{pmatrix}$$

and so the trace is

$$2\,\mathrm{tr}(A_1A_2) + \mathrm{tr}(B_1C_2) + \mathrm{tr}(C_1B_2)$$

If $\mathrm{tr}(xy) = 0$ for all $y \in \mathfrak{sp}_{2n}$, then in particular (taking $B_2 = C_2 = 0$), we have

$$2\,\mathrm{tr}(A_1A_2) = 0$$

for all $A_2 \in \mathfrak{gl}_n$. Then $A_1 = 0$. Taking $A_2 = B_2 = 0$ and allowing C_2 to run over a basis of symmetric matrices, we find $B_1 = 0$ and similarly $C_1 = 0$. Thus $x = 0$. Hence, $(-,-)$ is non-degenerate, and $\mathfrak{sp}_{2n}$ is semisimple.

For $\mathfrak{u}(n)$, we identify these with elements $x \in \mathfrak{gl}_{2n}(\mathbb{R})$ of the form

$$x = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

for $B \in \mathfrak{gl}_n(\mathbb{R})$ and $A \in \mathfrak{so}_n(\mathbb{R})$. The product of two such matrices is

$$\begin{pmatrix} A_1A_2 - B_1B_2 & A_1B_2 + B_1A_2 \\ -B_1A_2 - A_2B_1 & A_1A_2 - B_1B_2 \end{pmatrix}.$$

The trace is given by

$$\mathrm{tr}(xy) = 2\,\mathrm{tr}(A_1A_2) - 2\,\mathrm{tr}(B_1B_2)$$

If $\mathrm{tr}(xy) = 0$ for all y, then (taking $B_2 = 0$), non-degeneracy in the case of $\mathfrak{so}_n$ implies $A_1 = 0$. Similarly, taking $A_2 = 0$, then non-degeneracy in the case of $\mathfrak{gl}_n$ implies $B_1 = 0$. Hence $x = 0$, $(-,-)$ is non-degenerate and $\mathfrak{u}(n)$ is semisimple.

Exercise 15

i) Using freely the properties of the Killing form given below, show that

$$\mathfrak{so}_4(\mathbb{C}) = \{x \in \mathfrak{gl}_4(\mathbb{C}) \mid x + x^T = 0\}$$

is semisimple.

Hint: The Lie algebra has basis $F_{12}, F_{13}, F_{14}, F_{23}, F_{24}, F_{34}$ where F_{ij} has (k,ℓ) entry 1 if $i = k, j = \ell$; -1 if $i = \ell, j = k$; and 0 otherwise. The Killing form satisfies

$$\kappa(X,Y) = 2\,\mathrm{tr}(XY)$$

where XY is the product of X and Y as matrices.

ii) An ideal of the Lie algebra $\mathfrak{so}_4(\mathbb{C})$ is

$$I = \{\alpha A + \beta B + \gamma C \mid \alpha, \beta, \gamma \in \mathbb{C}\}$$

where

$$\begin{aligned} A &= i(F_{12} + F_{34}), \\ B &= \frac{i}{2}\left(F_{13} - F_{24} + iF_{14} + iF_{23}\right), \\ C &= \frac{i}{2}\left(F_{13} - F_{24} - iF_{14} - iF_{23}\right). \end{aligned}$$

Determine $I^{\perp}$ with respect to the Killing form.

Fun Fact! The only 3-dimensional simple Lie algebra over $\mathbb{C}$ is $\mathfrak{sl}_2(\mathbb{C})$, so we can conclude $I \cong I^{\perp} \cong \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$

Solution 15

i) To show that $\mathfrak{so}_4(\mathbb{C})$ is semisimple, it is sufficient to show the Killing form is non-degenerate. Using that

$$\operatorname{tr}(E_{i,j}E_{k,\ell}) = \begin{cases} 0 \text{ if } (i,j) \neq (j,i), \\ 1 \text{ otherwise.} \end{cases}$$

we compute that the matrix of the Killing form is $-4I$, which is invertible. Hence the Killing form is non-degenerate.

ii) Suppose

$$v = aF_{12} + bF_{13} + cF_{14} + dF_{23} + eF_{24} + fF_{34}$$

is orthogonal to I. Then, we must have

$$\begin{aligned} \kappa(v, A) &= -4i(a + f) = 0, \\ \kappa(v, B) &= -2i(b - e + ic + id) = 0, \\ \kappa(v, C) &= -2i(b - e - ic - id) = 0. \end{aligned}$$

This is equivalent to

$$a = -f,\ b = e,\ c = -d$$

and so

$$I^{\perp} = \{\alpha X + \beta Y + \gamma Z \mid \alpha, \beta, \gamma \in \mathbb{C}\}$$

where

$$\begin{aligned} X &= F_{12} - F_{34}, \\ Y &= F_{13} + F_{24}, \\ Z &= F_{14} - F_{23}. \end{aligned}$$