MAU34103 - Lie algebras and Lie groups

Practice Sheet 2

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

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1 Logarithms and exponentials

Exercise 1 *Exponential equations*

In lectures, we saw that, for $x, y \in \mathfrak{gl}_n(\mathbb{R})$, $\exp(x)\exp(y) = \exp(x+y)$ if $[x, y] = 0$. Let's explore this further and see if we can start to determine a general formula for the product.

i) Let $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Compute $e^x e^y$ and e^{x+y}.

ii) Let $x, y \in \mathfrak{gl}_n(\mathbb{R})$ now be arbitrary. Suppose that

$$[x, [x, y]] = [y, [x, y] = 0$$

Compute $e^{tx} y e^{-tx}$.

Hint: This is $\mathrm{Ad}_{e^{tx}}(y)$*. Convert this to a computation with the differential*

iii) Hence show that

$$f(t) = e^{tx}e^{ty}, \quad \text{and} \quad F(t) = e^{tx+ty+\frac{t^2}{2}[x,y]}$$

satisfy the same differential equation. Conclude that

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]}$$

Remark. *This is a special case of the Baker-Campbell-Hausdorff formula, knowing in quantum mechanics as the disentangling theorem. The general formula is that* $e^x e^y = e^z$ *with*

$$z = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \cdots$$

It is not to hard to verify this up to any given length of brackets, by explicitly expanding out $e^{tx}e^{ty} = e^{tz}$*, but proving it in general requires either some clever combinatorics or clever uses of ODEs*

Solution 1

i) Since $x^2 = y^2 = 0$, we have $e^x = I + x$ and $e^y = I + y$. In constrast

$$(x+y)^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n \text{ is even,} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } n\text{is odd.} \end{cases}$$

Thus, looking at the series expansion, we find

$$e^{x+y} = \begin{pmatrix} \cos(1) & \sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$$

while

$$e^x e^y = I + x + y + xy = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

ii) As noted, this is $\mathrm{Ad}_{e^{tx}}(y)$, which from a corollary in class is

$$e^{\mathrm{ad}_{tx}}(y) = y + t\,\mathrm{ad}_x(y) + \frac{t^2}{2}\,\mathrm{ad}_x(\mathrm{ad}_x(y)) + \cdots$$

But $\mathrm{ad}_x(\mathrm{ad}_x(y)) = [x,[x,y]] = 0$, and so

$$e^{tx}ye^{-ty} = e^{\mathrm{ad}_{tx}}y) = y + t[x,y]$$

iii) We compute

$$f'(t) = xe^{tx}e^{ty} + e^{tx}ye^{ty} = (x + e^{tx}ye^{-tx})e^{tx}e^{ty} = (x + y + t[x,y])f(t)$$

and

$$F'(t) = (x + y + t[x,y])e^{tx+ty+\frac{t^2}{2}[x,y]} = (x + y + t[x,y])F(t).$$

Thus $f(t)$ and $F(t)$ satisfy the same ODE. As $f(0) = I = F(0)$, we therefore have $f(t) = F(t)$ for all t. In particular, $f(1) = F(1)$ and so

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]}$$

Exercise 2 *A logarithmic failure*

Recall that, for a linear group G, $x \in L(G)$ if $e^{tx} \in G$ for all $t \in \mathbb{R}$. We will see that it is not sufficient to have just $e^x \in G$.

i) Determine non-zero $x \in \mathfrak{gl}_2(\mathbb{R})$ such that $e^x = I$

Hint: In the complex numbers $e^{2\pi i} = 1$. Do we have a map $f : \mathbb{C} \to \mathfrak{gl}_2(\mathbb{R})$ such that $f(e^z) = e^{f(z)}$?

ii) Given an example of a linear group $G \subset \mathrm{GL}_2(\mathbb{R})$ such that $I \in G$, but $x \notin L(G)$ for this x.

Hint: Think lower dimensional

Solution 2

i) In the first homework, we saw an $\mathbb{R}$-linear map

$$f(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

such that $f(zw) = f(z)f(w)$. Thus

$$f(e^z) = \sum_{n\geq 0} \frac{f(z^n)}{n!} = \sum_{n\geq 0} \frac{f(z)^n}{n!} = e^{f(z)}.$$

Thus, as $e^{2\pi i} = 1$, we have

$$e^{f(2\pi i)} = f(e^{2\pi i}) = f(1) = I$$

but $f(2\pi i) \neq 0$.

ii) We can take G to be the linear group of diagonal matrices, of upper triangular matrices, or even $G = \{I\}$.

Exercise 3 *1-parameter subgroup (or general exponentiation)*

Given a Lie group G, we call a Lie group homomorphism $\gamma : \mathbb{R} \to G$ (and its image) a *1-parameter subgroup*. Let $\mathfrak{g} = T_I G$ be the Lie algebra given by the tangent space to the identity. Note that, by choosing a chart around I, we may assume $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ for some n. Given a chart (U, ϕ) you may also assume that $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in U$ such that $gh \in U$.

i) If G is a linear group, and $x \in L(G)$, show that $t \to e^{tx}$ is a 1-parameter subgroup of G.

ii) Show that a 1-parameter subgroup is determined by $\gamma(t)$ for $t \in [0, n^{-1})$ for any $n \geq 1$.

iii) Show that any 1-parameter subgroup satisfies a differential equation near $t = 0$, given in a chart (U, ϕ) around I by

$$(\phi \circ \gamma)'(t) = x(\phi \circ \gamma)(t)$$

for some $x \in \mathfrak{g}$.

Hint: Differentiate the definition of a homomorphism

iv) Show that, for any $x \in L(G)$, a solution to

$$(\phi \circ \gamma)'(t) = x(\phi \circ \gamma)(t)$$

in a chart around I satisfies

$$\gamma(s+t) = \gamma(s)\gamma(t)$$

for $s, t \in (-\varepsilon, \varepsilon)$ for some sufficiently small $\varepsilon > 0$.

Hint: A differential equation of this form has a unique solution given fixed initial conditions

v) Hence conclude that, for any tangent vector x, there is a unique 1-parameter subgroup γ such that $\gamma'(0) = x$ (in some chart).

Such subgroups are how we define $\exp : \mathfrak{g} \to G$ in general. Namely, we define $\exp(x) := \gamma_x(1)$, where $\gamma_x(t)$ is the unique 1-parameter subgroup determined by x.

vi) Show that $\exp((s+t)x) = \exp(sx)\exp(tx)$.

Hint: Chain rule and uniqueness of solutions to ODEs should let you compute $\exp(tx)$ *explicitly in terms of* γ_x.

Remark. *For those of you more familiar with differential geometry, a tangent vector $x \in T_I G$ determines a vector field X such that $X_g(f) = X(f \circ g)$ for all $g \in G$ and $f : G \to \mathbb{R}$. Such a vector field is called left-invariant, as they are invariant under the push forward associated to the left multiplication map $L_g : G \to G$. The Peano Existence Theorem implies these is a unique integral curve γ through I associated to X. As, for any given s, $\gamma(s+t)$ and $\gamma(s)\gamma(t)$ are both integral curves through the same point, they must be equal, and so the integral curve is a 1-parameter subgroup.*

Solution 3

i) Since $[sx, tx] = 0$ for all $s, t \in \mathbb{R}$, we have that

$$e^{(s+t)x} = e^{sx}e^{tx}$$

and so e^{tx} defines a 1-parameter subgroup.

ii) Suppose $t \notin [0, n^{-1})$. If $t < 0$, then we must have $\gamma(t) = \gamma(-t)^{-1}$, so it suffices to consider $t \geq n^{-1}$. For any $t > 0$, there exists an integer $N > 0$ such that

$$\frac{t}{N} < \frac{1}{n}$$

and so $\gamma\left(\frac{t}{N}\right)$ is known. As γ is a 1-parameter subgroup, we must have

$$\gamma(t) = \gamma\left(\frac{t}{N}\right)^N$$

and so we can determine $\gamma(t)$ for all $t \in \mathbb{R}$.

iii) We have

$$(\phi \circ \gamma)(s+t) = \phi(\gamma)(s)\gamma(t)) = (\phi \circ \gamma)(s)(\phi \circ \gamma)(g)$$

for all s, t such that all group elements Lie within U. We can ensure this for all small enough s, t thanks to the smoothness of γ and multiplication. Differentiating this with respect to s and evaluating at $s = 0$, we find

$$(\phi \circ \gamma)'(t) = (\phi \circ \gamma)'(0)(\phi \circ \gamma)(t)$$

for all small enough t. As $(\phi \circ \gamma)'(0)$ is a tangent vector (in this chart), it defines an element $x \in \mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$.

iv) We will omit reference to the chart to reduce notation and note that, for any fixed t, both

$$f(s) = \gamma(s+t) \quad \text{and} \quad g(s) = \gamma(s)\gamma(t)$$

satisfy

$$f'(s) = xf(s) \quad \text{and} \quad g'(s) = xg(s).$$

As $f(0) = \gamma(t) = g(0)$, we must have $f(s) = g(s)$ for all sufficiently small s. Hence, the claim holds.

v) Given a tangent vector, we can find some n and a smooth $\gamma : (-n^{-1}, n^{-1}) \to G$ such that

$$\gamma(s+t) = \gamma(s)\gamma(t)$$

for all sufficiently small s, t and $\gamma'(0) = x$ in some chart, by solving the ODE

$$\gamma'(t) = x\gamma(t).$$

As a 1-parameter subgroup is determined by its values on $[0, n^{-1})$ for any n, this allows us to uniquely extend to a function $\gamma : \mathbb{R} \to G$ as in part (ii), assuming $\gamma(t)$ is independent of which N we choose. But, if

$\frac{t}{N}, \frac{t}{M} \in (-n^{-1}, n^{-1})$, then $\frac{t}{MN} \in (-n^{-1}, n^{-1})$. As γ is a homomorphism near 0, we have

$$\gamma\left(\frac{t}{N}\right) = \gamma\left(\frac{t}{MN}\right)^M$$

and

$$\gamma\left(\frac{t}{M}\right) = \gamma\left(\frac{t}{MN}\right)^N$$

so we have

$$\gamma\left(\frac{t}{N}\right)^N = \gamma\left(\frac{t}{MN}\right)^{MN} = \gamma\left(\frac{t}{M}\right)^M$$

and can therefore define $\gamma(t)$ uniquely.

2 Induced algebra homomorphisms

Exercise 4 *A complex correspondence*

Consider the Lie groups

$$U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$$
$$\mathrm{SO}(2) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid A^T A = I\}$$

i) Show that the Lie algebra (over $\mathbb{R}$) of $U(1)$ is

$$\mathfrak{u}(1) = \{z \in \mathbb{C} \mid z + \overline{z} = 0\}.$$

ii) Show that the map

$$U(1) \to \mathrm{SO}(2)$$
$$e^{it} \mapsto \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

is an isomorphism.

iii) Determine the induced isomorphism

$$\mathfrak{u}(1) \to \mathfrak{so}_2(\mathbb{R}).$$

Solution 4

i) As we have a map taking complex exponentials to matrix exponentials, we can say that

$$\begin{aligned}\mathfrak{u}(1) &= \{z \in \mathbb{C} \mid e^z \in U(1)\} \\ &= \{z \in \mathbb{C} \mid |e^z| = 1\} \\ &= \{z \in \mathbb{C} \mid Re(z) = 0\} \\ &= \{z \in \mathbb{C} \mid z + \overline{z} = 0\}.\end{aligned}$$

ii) It is clearly injective, as if $(\cos(t), \sin(t)) = (\cos(s), \sin(s))$, then

$$e^{it} = \cos(t) + i\sin(t) = \cos(s) + i\sin(s) = e^{is}.$$

To see that it is surjective, suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SO(2)$. We must have that

$$ad - bc = 1, \quad a^2 + c^2 = b^2 + d^2 = 1, \quad ac + bd = 0.$$

The middle equality implies that

$$a = \cos(\theta),\ c = \sin(\theta),\ d = \cos(\phi),\ b = \sin(\phi)$$

for some angles θ, ϕ. The third equality implies that

$$\sin(\theta + \phi) = 0$$

and so we have that $\theta + \phi = \pi k$ for some integer k. From the determinant equality, we have

$$\cos(\theta)\cos(k\pi - \theta) - \sin(\theta)\sin(k\pi - \theta) = 1$$

and so

$$(-1)^k \left(\cos^2(\theta) + \sin^2(\theta)\right) = 1$$

Thus k is even, and so we can take it to be 0 thanks to the periodicity of the trig functions. Thus $\phi = -\theta$ and so every element of SO(2) is the image of e^{it} for some t.

Hence the map is surjective and therefore an isomorphism.

iii) The differential df of a map f satisfies that

$$df(x) = \frac{d}{dt}|_{t=0} f(e^{tx})$$

In our case, we have

$$df(ix) = \frac{d}{dt}|_{t=0} \begin{pmatrix} \cos(tx) & -\sin(tx) \\ \sin(tx) & \cos(tx) \end{pmatrix} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$$

which is indeed an isomorphism.

Exercise 5 *An induced isomorphism*

Consider the homomorphism of Lie groups

$$\begin{aligned} f : (\mathbb{R}, +) &\to (U(1), \cdot) \\ t &\mapsto e^{it}. \end{aligned}$$

i) Explain, without any computation, why the induced Lie algebra homomorphism must be an isomorphism.

ii) Determine explicitly the induced Lie algebra homomorphism.

iii) Determine $\ker(f)$ and based on its topological properties, make a wild conjecture about when two Lie groups will have the same Lie algebra.

Solution 5

i) Both Lie group have dimension 1 ($U(1)$ is just the circle), and so their Lie algebras are 1-dimensional, and all 1-dimensional Lie algebras are isomorphic.

ii) Recall that

$$(\mathbb{R}, +) \cong \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

Using the isomorphism $U(1) \to \mathrm{SO}(2)$, the homomorphism is equivalent to the map

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}$$

and so

$$df(x) = \frac{d}{dt}|_{t=0} \begin{pmatrix} \cos(tx) & -\sin(tx) \\ \sin(tx) & \cos(tx) \end{pmatrix} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$$

or, in the original groups

$$df(x) = ix.$$

iii) The kernel of f is given by

$$\ker f = \{2\pi ik \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$$

which is a discrete subgroup of $\mathbb{R}$. Let's wildly conjecture that two Lie gorups have the same Lie algebra if one is a quotient of the other by a discrete subgroup.

3 Subgroups and quotient groups

The following questions increasingly lean into the geometry of Lie groups, which we have not spent much time on, and so should be considered on the more challenging side

Exercise 6 *A local almost group*

Let G be a Lie group with identity id. We will show that, for any open $U \ni \text{id}$, there exists an open $V \subset U$ containing id such that

$$V = V^{-1} = \{g^{-1} \mid g \in V\}, \quad V^2 = \{gh \mid g, h \in V\} \subset U.$$

i) Show that, for any open $U \subset G$, the set

$$U^{-1} = \{g^{-1} \mid g \in Y\}$$

is open. Show that, if $U \ni \text{id}$, then there exists open $V \subset U$ containing id such that $V = V^{-1}$.

Hint: The map $g \to g^{-1}$ is smooth and hence continuous.

ii) Show that, for any open $U \subset G$ containing the identity, there exist open $V_1, V_2 \subset G$ containing the identity such that

$$V_1V_2 := \{gh \mid g \in V_1,\ h \in V_2\} \subset U.$$

Hint: The map $(g, h) \mapsto gh$ is continuous for the product topology. Be careful to remember how the product topology is defined!

iii) By taking appropriate intersections, show there exists an open $V \subset U$ containing the identity such that

$$V = V^{-1}, \quad , V^2 \subset U.$$

Solution 6

i) The map $\iota : g \mapsto g^{-1}$ is smooth, and hence continuous. Thus, the inverse image of any open set is open. But

$$U^{-1} = \iota^{-1}(U)$$

so U^{-1} is open.

Let $V = U \cap U^{-1}$. This is open, as it is an intersection of two opens, and

$$V^{-1} = \iota(V) = \iota(U) \cap \iota(U^{-1}) = U^{-1} \cap U = V$$

The identity is in V as $\mathrm{id} \in U$ and $\mathrm{id} = \mathrm{id}^{-1} \in U^{-1}$.

ii) As multiplication $m : G \times G \to G$ is continuous $m^{-1}(U)$ is an open set in $G \times G$. Since $m(\mathrm{id}, \mathrm{id}) = \mathrm{id} \in U$, $(\mathrm{id}, \mathrm{id}) \in m^{-1}(U)$. As products of open sets in G form a base for the product topology, there must be open $V_1, V_2 \subset G$ containing id such that

$$V_1 \times V_2 \subset m^{-1}(U)$$

Hence

$$V_1 V_2 = m(V_1 \times V_2) \subset U$$

iii) As $\mathrm{id} \in V_1, V_2$, $\tilde{V} = V_1 \cap V_2$ is a non-empty open set such that $\tilde{V}^2 \subset U$. Let $V = \tilde{V} \cap \tilde{V}^{-1}$. From the previous parts, we have that V is a non-empty open set such that $V = V^{-1}$ and $V^2 \subset \tilde{V}^2 \subset U$ as needed.

Exercise 7 *Actions of discrete subgroups*

We call a normal subgroup $H \subset G$ of a Lie group *discrete* if there exists an open set $U \ni \mathrm{id}$ such that

$$H \cap U = \mathrm{id}$$

We say that H acts *properly discontinuously* on G if, around every $g \in G$, there is an open set $U_g \ni g$ such that

$$hU_g \cap U_g = \emptyset \quad \text{for all} \quad \mathrm{id} \neq h \in H,\ g \in G$$

i) Show that a subgroup H acts properly discontinuously on G if there exists an open $U \ni \mathrm{id}$ such that

$$hU \cap U = \emptyset \quad \text{for all} \quad \mathrm{id} \neq h \in H$$

ii) Hence conclude that a discrete normal subgroup acts properly discontinuously on G.

Hint: Consider the open set from the previous problem

Solution 7

i) If such a U exists, we can take $U_g = gU$. As multiplication by g is a homeomorphism, this is an open set and

$$hU_g \cap U_g = g\left(g^{-1}hgUcapU\right) = g\emptyset = \emptyset$$

for all $h \in H$.

ii) Let U be an open set around id such that $H \cap U = \{\text{id}\}$ and let V be as in the previous exercise. We claim $hV \cap V = \emptyset$ for all $h \neq \text{id}$ in H.

Suppose $g \in hV \cap V$ for some $h \in H$. Then, as $g \in hV$, we have

$$g = hs$$

for some $s \in V$. From the properties of V, $s^{-1} \in V$ and so $h = gs^{-1} \in U$. But $H \cap U = \{\text{id}\}$, so $h = \text{id}$. Thus $hV \cap V = \emptyset$ for all $h \neq \text{id}$.

Exercise 8 *Discrete subgroups are closed*

A subgroup $H \subset G$ is called discrete if there exists an open set $U \ni \text{id}$ such that

$$H \cap U = \{\text{id}\}$$

Let us show H contains all its limit points in G and is therefore closed.

i) Suppose we have a sequence $\{h_n\} \subset H$ such that $\lim_{n\to\infty} h_n = g \in G$. Show that the sequence

$$x_n = h_n h_{n+1}^{-1}$$

defines a sequence in H such that $\lim_{n\to\infty} x_n = \text{id}$.

ii) Hence conclude that $x_n = \text{id}$ for all $n \geq N$ for some n.

Hint: Discreteness and closeness are incompatible

iii) Hence conclude that $g \in H$.

Solution 8

i) Well, $x_n \in H$ as H is a subgroup. Since the sequence h_n converges, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} j_n \lim_{n\to\infty} h_{n+1}^{-1} = gg^{-1} = \mathrm{id}\,.$$

ii) As H is discrete, there is some open set U around id such that $H \cap U = \{\mathrm{id}\}$. As $x_n \to \mathrm{id}$, we must have that there exist $N \geq 0$ such that $x_n \in U$ for all $n \geq N$. So for all $n \geq N$, $x_n \in H \cap U = \{\mathrm{id}\}$, i.e. $x_n = \mathrm{id}$.

iii) Hence, for all $n \geq N$, $h_n = h_{n+1}$ and so

$$h_N = h_{N+1} = h_{N+2} = \cdots$$

from which is it clear that $g = \lim_{n\to\infty} h_n = h_N \in H$.

Exercise 9 *Quotient Lie groups*

The Quotient Manifold theorem states that, if H is a Lie group acting smoothly on a manifold M, then the orbit space M/H is a manifold if

- H acts freely on M: if $h \cdot x = x$ for any $x \in M$, then $h = \mathrm{id}$.
- H acts properly on M: if K_1 and K_2 are compact subsets of M, then the set

  $$\kappa := \{h \in H \mid h \cdot K_1 \cap K_2 \neq \emptyset\}$$

  is compact.

A closed normal subgroup $H \subset G$ of a Lie group acts smoothly on G by left multiplication

$$(h, g) \mapsto hg.$$

i) Show that H acts freely on G.

Hint: No need for Lie theory here

ii) Show that

$$\kappa = H \cap K_2 K_1^{-1}$$

where

$$K_2 K_1^{-1} = \{g_2 g_1^{-1} \mid g_1 \in K_1,\ g_2 \in K_2\}$$

iii) Hence show that a closed subgroup H acts properly on G.

Hint: Why is $K_2K_1^{-1}$ compact? H is closed, so that would be enough.

iv) Conclude that G/H is both a group and a manifold for H a closed normal subgroup.

It is in fact a Lie group, but showing the group operations are smooth requires more differential geometry than we have time for

Solution 9

i) If $hg = g$ for any $g \in G$, then multiplying by g^{-1} implies $h = \mathrm{id}$.

ii) Suppose $h \in \kappa$. Then, there exists $g_1 \in K_1$ and $g_2 \in K_2$ such that $hg_1 = g_2$, and so $h = g_2 g_1^{-1}$. Thus

$$\kappa \subset H \cap K_2K_1^{-1}$$

Conversely, if $h \in H \cap K_2K_1^{-1}$, then there exist $g_i \in K_i$ such that

$$h = g_2g_1^{-1} \quad \Rightarrow hg_1 = g_2$$

and hence $hK_1 \cap K_2 \neq \emptyset$. Thus

$$\kappa \supset H \cap K_2K_1^{-1}$$

iii) We have a continuous map

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\mapsto gh^{-1} \end{aligned}$$

that sends $K_2 \times K_1 \to K_2K_1^{-1}$. Since $K_2 \times K_1$ is compact in the product topology, and the continuous image of a compact set is compact, $K_2K_1^{-1}$ is compact.

As H is closed (and G is Hausdorff), the intersection $H \cap K_2K_1^{-1}$ is also compact. Hence H acts properly on G.

iv) G/H is a group as H is normal, and the Quotient Manifold Theorem tells us it is a manifold as H is closed.